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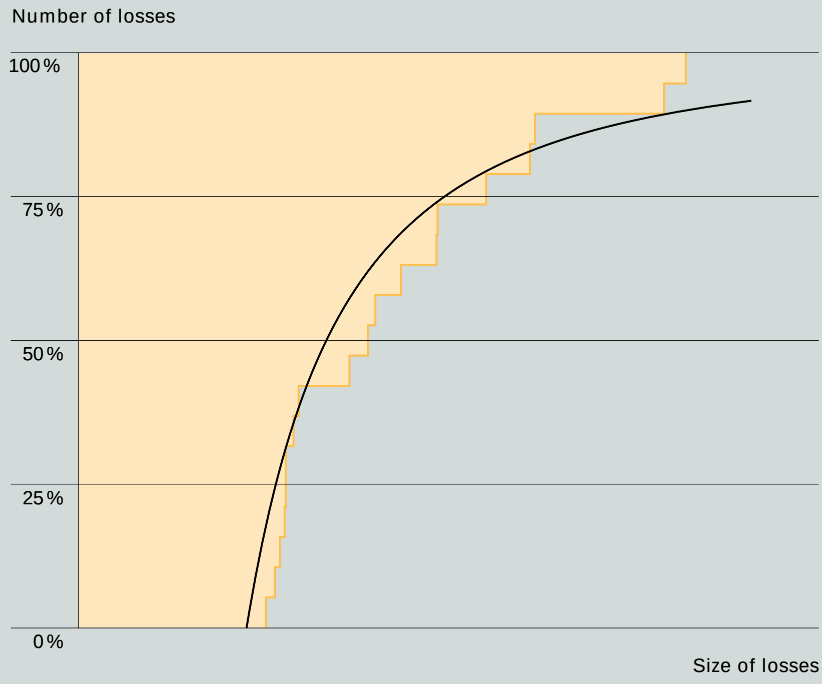

Estimating
property excess of loss risk premiums
by means of the Pareto model

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Hans Schmitter
Revised by Peter Bütikofer

Foreword

Hans Schmitter's brochure “Property excess loss rating by means of the Pareto model” was first published in 1978. His presentation of this complex subject proved so practical that the original brochure was reprinted several times over subsequent years. However, the introduction of the new Swiss Re corporate identity meant that this was no longer possible. Thus, almost twenty years after the document was first published, we seized the opportunity to revise it somewhat. The current publication retains the majority of Hans Schmitter's original ideas.

In addition to certain supplementary observations, we have included a chapter dedicated to “the price of reinsurance”. Moreover, various alterations have been undertaken in order to conform to standard corporate abbreviations and terms to be used in all future Swiss Re publications.

Peter Bütikofer, November 1997

1 Introduction

An underwriter has two methods at his disposal for estimating the risk premium (= expected average loss burden) for reinsurance treaties: past losses (experience rating) or current portfolio characteristics such as risk exposure and composition (exposure rating). As a rule he will exploit all available means in order to compose a reliable estimate of the risk premium.

The Pareto model can be used in conjunction with both experience and exposure rating, depending on the origin of the losses used to calculate the model's parameters. If the underwriter uses only losses incurred in the portfolio to be reinsured, this is known as experience rating. Yet he will also have experience values for parameters of specific portfolios. If he considers the characteristics of the portfolio in order to adjust these experience values, then he uses the model for exposure rating.

The Pareto model is often used to estimate risk premiums for excess of loss treaties with high deductibles, where loss experience is insufficient and could therefore be misleading. The relevant procedures are detailed in sections 7 to 10. This brochure is, however, intended to go beyond a mere collection of "recipes" and provide the reader with an insight into the basic assumptions that lead to a mathematical model for excess loss rating.

2 Why a model?

Estimating the risk premium of an excess loss treaty is simple, provided that the excess losses are frequent, affect the entire range of cover and thus lead to a regular burning cost*. However treaties of this type are rare because a treaty with a relatively stable burning cost hardly needs to be reinsured. If the protected portfolio grows, the burning cost becomes more regular and the fluctuation potential declines, so that the ceding company can and will cut down the reinsurance premiums by raising the deductible. The reinsurer is thus rarely able to rate risk premiums for excess loss treaties using only reliable loss experience. Usually, with little or no loss experience, he must rely either upon losses which are below the deductible (experience rating) and/or upon his knowledge of similar situations (exposure rating).

A mathematical model is nothing more than an abstraction and simplification of comprehensive experience. A simple and useful one is known as the Pareto model. Although actuaries have recently proposed a range of alternative models which permit more accurate calculation, these models are more complicated and hinder the development of an instinctive feel for the parameters. The Pareto model is thus likely to remain the most important mathematical model for calculating property excess of loss premiums for some years to come.

*Definition of "burning cost": Method of calculating premiums in reinsurance in which the reinsurance premium is determined independently of the direct insurer's original premium. Data from the past is used to calculate the hypothetical loss burden (burning cost) which the reinsurer would have had to carry over the last three to five years if he were to have concluded a treaty with the insurer in question at that time. Next, the value thus obtained is extrapolated into the future, taking into account possible changes (eg inflation, portfolio changes).

3 Graphical representation of losses

Throughout this publication we will use figures taken from the four years' loss experience of one of our Italian ceding companies as an illustrative example. Our rating will be based on the first three years and will subsequently be compared to the actual experience in the fourth year. The losses are listed in tables 1 to 4. The currency unit is ITL 1000.

Table 1

Losses in year 1 (index 110.6)

Critical deductible[1]: 46471

Index 110.6	Index 119.0 (projection to 4th year)
60800.0	65417.7
55640.0	59865.8
98800.0	106303.8
54250.0	58370.3
167700.0	180436.7
80000.0	86075.9
112520.0	121065.8
126000.0	135569.6

Table 2

Losses in year 2 (index 113.2)

Critical deductible: 47563

Index 113.2	Index 119.0 (projection to 4th year)
91000.0	95662.5
165500.0	173979.7
101460.0	106658.5
76500.0	80419.6
53000.0	55715.5

[1] We will concern ourselves only with those losses which would have exceeded a deductible of 50000, were they to have occurred in year 4. From year 1 (index 110.6) this would therefore include all losses over 46471, since

$$46471 \times \frac{119.0}{110.6} = 50000.$$

These losses are listed in table 1. 46471 is the so-called “critical deductible”.

Table 3

Losses in year 3 (index 117.4)

Critical deductible: 49 328

Index 117.4	Index 119.0 (projection to 4th year)
60 690.0	61 517.1
60 380.0	61 202.9
60 690.0	61 517.1
87 000.0	88 185.7
63 000.0	63 858.6
132 200.0	134 001.7

Table 4

Losses in year 4 (index 119.0)

Index 119.0
84 000.0
148 050.0
84 090.0
66 500.0
66 400.0
177 550.0
71 000.0
85 000.0
61 000.0
63 100.0
58 000.0

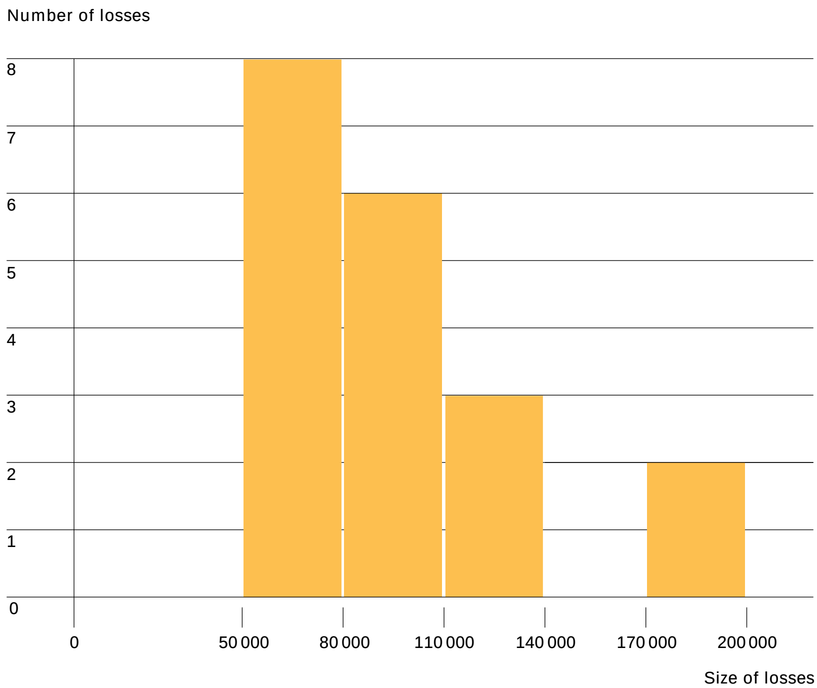
Using the indexed losses from the first three years, we could for example establish five classes:

50 000 – 79 999 (8 losses)
80 000 – 109 999 (6 losses)
110 000 – 139 999 (3 losses)
140 000 – 169 999 (0 losses)
170 000 – 199 999 (2 losses)

Such an arbitrary classification would give us the following figure 1.

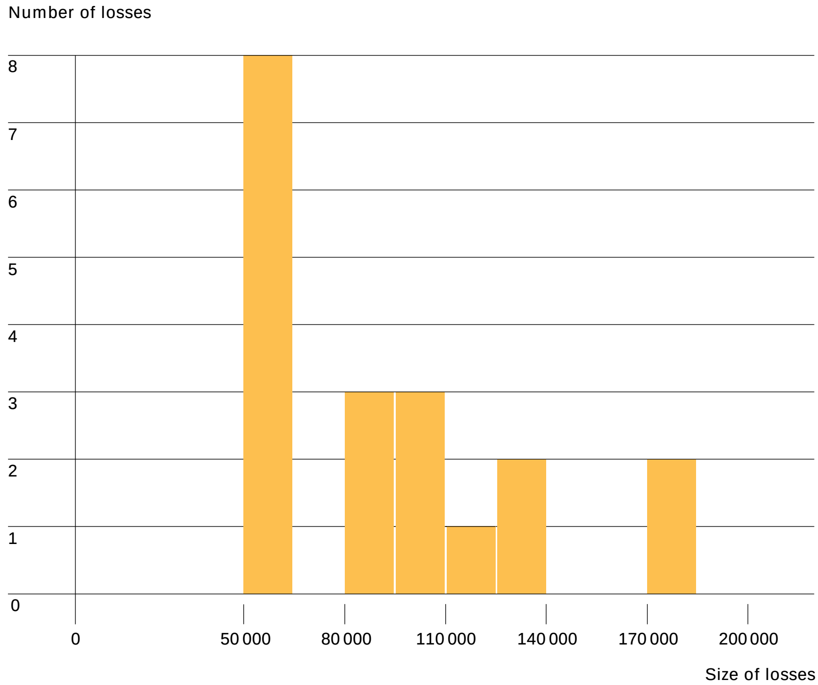
Figure 1 represents the distribution of the indexed losses in the years 1 to 3.

Figure 1



Apparently, the appearance of the graph is dependent on the choice of class size (30 000 in this example). If we choose smaller classes, eg 50 000–64 999, 65 000–79 999 etc, the distribution is as shown in figure 2.

Figure 2



A further reduction in class size would obviously lead to useless graphs with a large number of classes, each containing only one loss or none at all.

There is a superior method of graphical representation which is not dependent on a random choice of class size:

Losses are arranged by size and then put one on top of the other, starting with the smallest. We will use only this method from now on. Its result is the flight of stairs shown in figure 3.

Figure 3

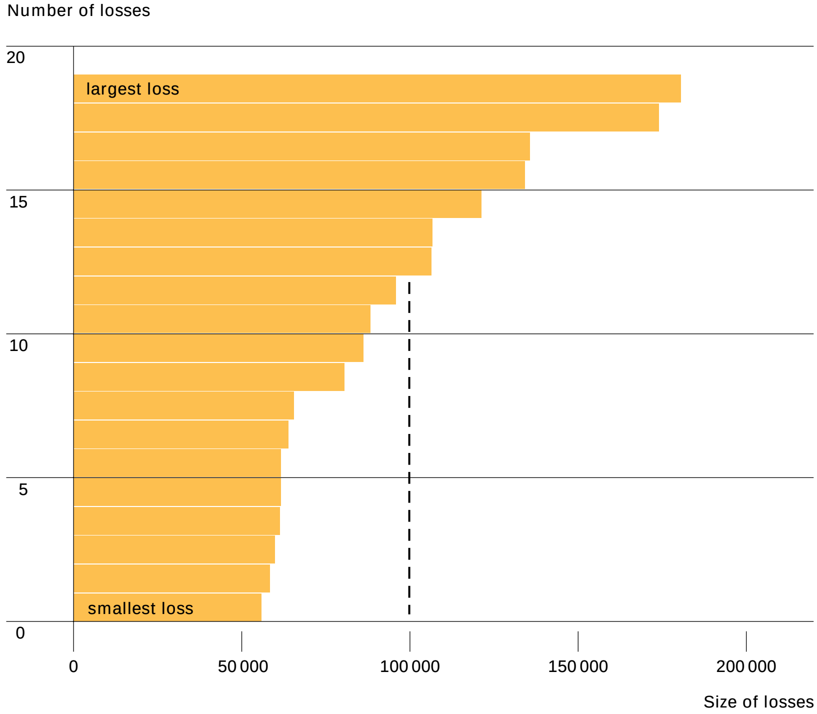
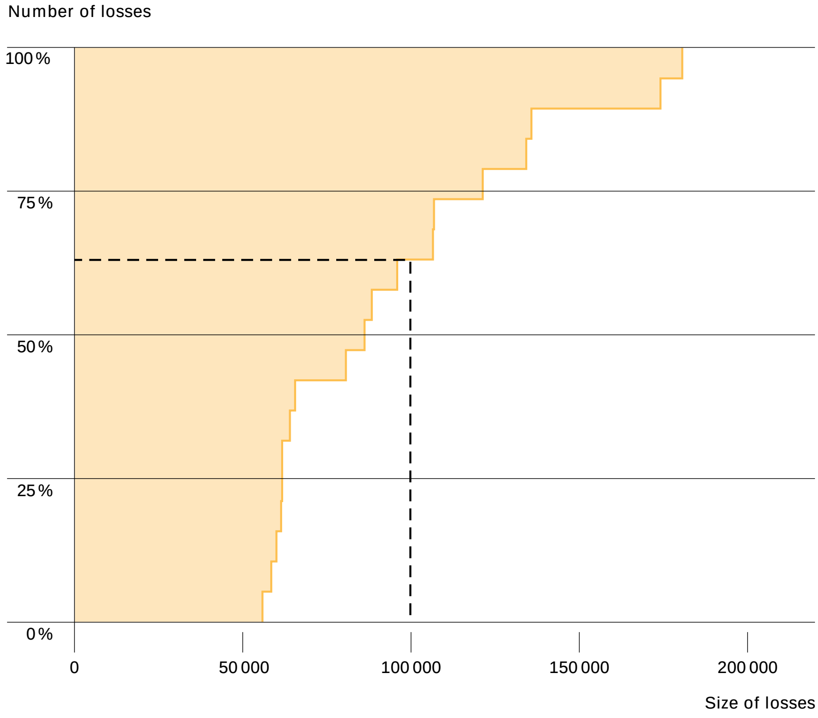


Figure 4 is the same as figure 3, but the scale of the vertical axis has been adjusted. Instead of counting from loss no. 1 to 19 we count from 0 to 100% of all losses by number. Graphs such as figure 4 are known as “empirical distribution functions”.

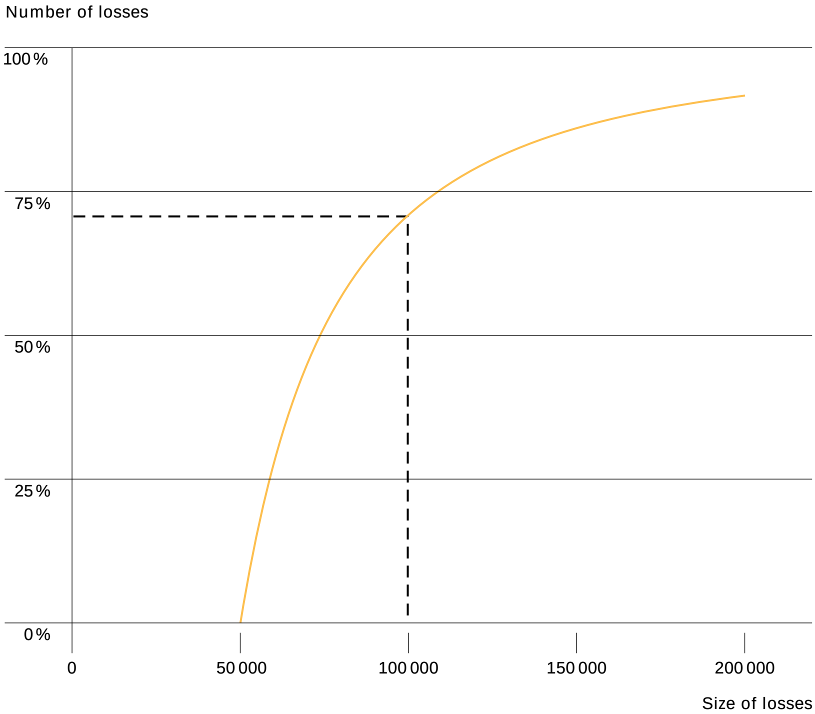
Figure 4



Graphs such as figures 3 and 4 show us how many claims remained below a certain level or just reached it, and how many exceeded it. Thus, according to figure 3, 12 out of 19 losses were smaller than 100 000 (see dashed line); figure 4, of course, shows exactly the same, ie that 63.16% of the total number of losses were smaller than 100 000.

If we had more extensive loss statistics, eg an experience of 100 losses, we could still represent them in a graph of the figure 4 type. The only difference would be that one single step would measure 1% of the total height instead of 5.26%. If our statistics consisted of an infinite number of losses, it would produce a smooth curve as shown in figure 5.

Figure 5



In theory, smooth curves such as the one shown in figure 5 could be drawn for different types of risks (residential property, industrial risks etc), but in practice this is neither possible nor necessary. For every type of risk, a graph with only about 100 stairs already roughly indicates the shape of the smooth curve which it would tend to become if the number of losses increased. A similar curve may then be described by a relatively simple mathematical model. Models of this kind are called “distribution functions” or “distributions”. One of the most elementary ones which is well-suited for property excess of loss rating is known as the Pareto distribution.

In order to draw a smooth, representative curve for a given “flight of stairs”, we need to know the starting point and the steepness of the stairs. In our example, we are concerned only with losses above 50 000. This is the starting point of the flight of stairs. The steepness of the stairs can be calculated from known losses (see appendix 2). In the Pareto model, this is shown by the exponent or Pareto parameter. In our example the parameter is 1.77.
The Pareto distribution describing a curve closest to the one in figure 5 has the form:

expected number of losses between 50 000 and x, divided by the total number of losses over 50 000

$$= 1 - \left[\frac{50\,000}{x}\right]^{1.77}$$

The expected number of losses of a particular size class (eg between 50 000 and 100 000), divided by the total number of expected losses is called “probability” (eg 0.71 or 71%). The above equation can therefore be rewritten as

probability of a loss between 50 000 and x

$$= 1 - \left[\frac{50\,000}{x}\right]^{1.77}$$

We should not forget that we know nothing about losses below 50 000. We can only comment on the probability of a loss exceeding 50 000 if we presume that the loss has already attained the 50 000 minimum. A probability of 71% for losses between 50 000 and 100 000 means therefore that 71% of all losses over 50 000 will not exceed 100 000.

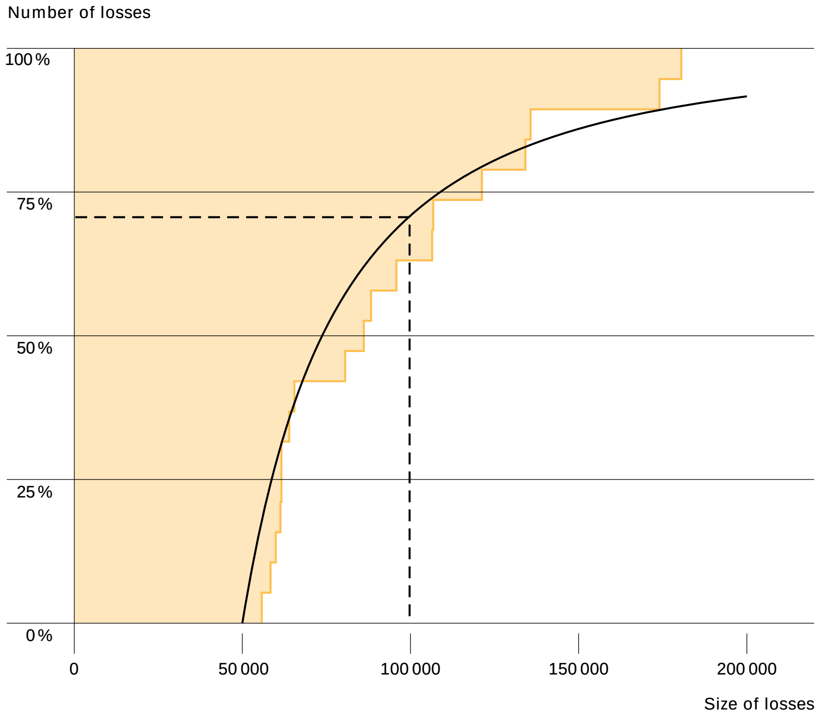
The left-hand side of the equation is usually abbreviated to $P(X \leqslant x)$ which means “probability that a loss is less than or equal to x”.[2] The Pareto distribution can thus be shortened to

$$P(X \leqslant x) = 1 - \left[\frac{50\,000}{x}\right]^{1.77}$$

Figure 6 shows the curve corresponding to the above Pareto distribution. The yellow flight of stairs is the same as in figures 3 and 4. Note that it differs from the smooth curve: as we have seen in section 3, it indicates that 63.16% of all losses by number remained below 100 000, while the curve shows that the probability of a loss being less than 100 000 is 71%.

[2] In the abbreviation $P(X \leqslant x)$, P stands for probability, X for loss and x for the amount which the loss should not exceed.

Figure 6



The numerator on the right-hand side of the equation, ie 50 000, defines the smallest loss taken into account by the Pareto distribution. The exponent, 1.77, is known as the “Pareto parameter”.

Instead of choosing 50 000 as the smallest loss, we could also select 80 000 or 100 000 or any other high value.[3] As far as excess loss rating is concerned, the smallest loss is often equivalent to the deductible or the lowest observation point from a list of losses.

Likewise, the Pareto parameter need not be 1.77. Any positive value defines a Pareto distribution. The general form of the Pareto distribution is usually written as

$$P(X \leqslant x) = 1 - \left[\frac{OP}{x}\right]^{\alpha}$$

where OP (observation point)[4] defines the smallest loss taken into consideration and α is the Pareto parameter. Experience has shown that the Pareto parameters describing the distribution of fire losses usually vary between 1.5 und 2.5. Parameters describing catastrophic losses (earthquake, wind storm, etc.) usually lie around 1 but can also be less (between 0 and 1). A method of determining the Pareto parameter from observed losses is described in appendix 2.

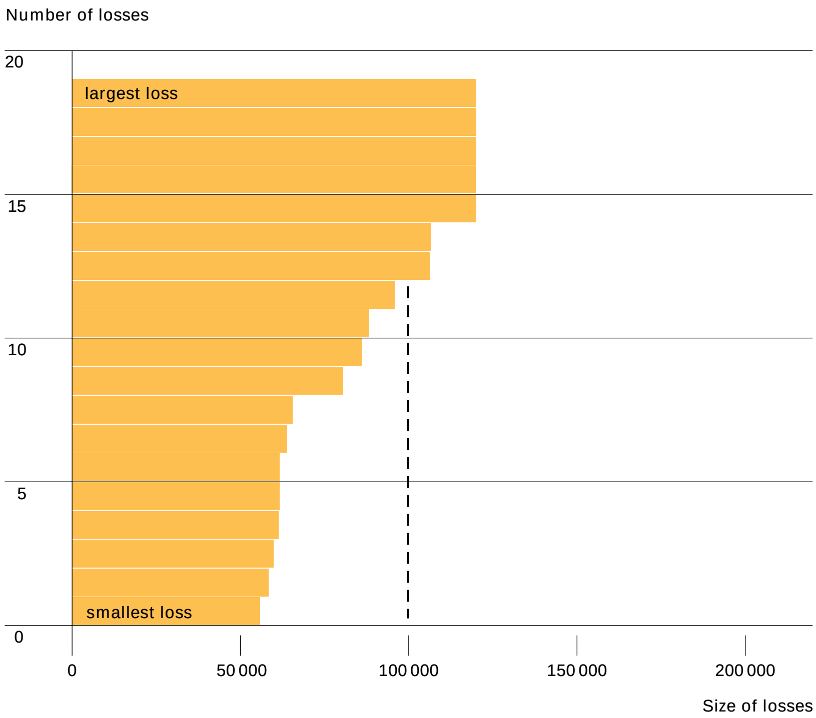
[3] Graphs of the type depicted by figures 3 and 4, which also include very small losses, are usually S-shaped. The smooth curves which such graphs tend to become as additional stairs are added would therefore also have to be S-shaped. However, as Pareto curves are convex (and can thus only show right-hand curves), they can only describe the part on the right of the turning point of the S-shaped curve. Therefore the smallest loss they take into consideration must itself lie on the right of the turning point, or, in other words, be sufficiently high.
Other distribution functions which describe an S-curve require at least 3 model parameters. However, due to a lack of experience for these model parameters, these distribution functions are infrequently applied in underwriting.

[4] All symbols and abbreviations used in this publication are given in appendix 3.

In sections 3 and 4 we introduced the loss distribution function as a graphical representation of losses. If we truncate every loss at a certain limit, say 120 000, we must modify our graphs accordingly. Such truncations are the effect of cover limits as used in property excess of loss reinsurance. The resulting functions are known as “truncated distribution functions”.

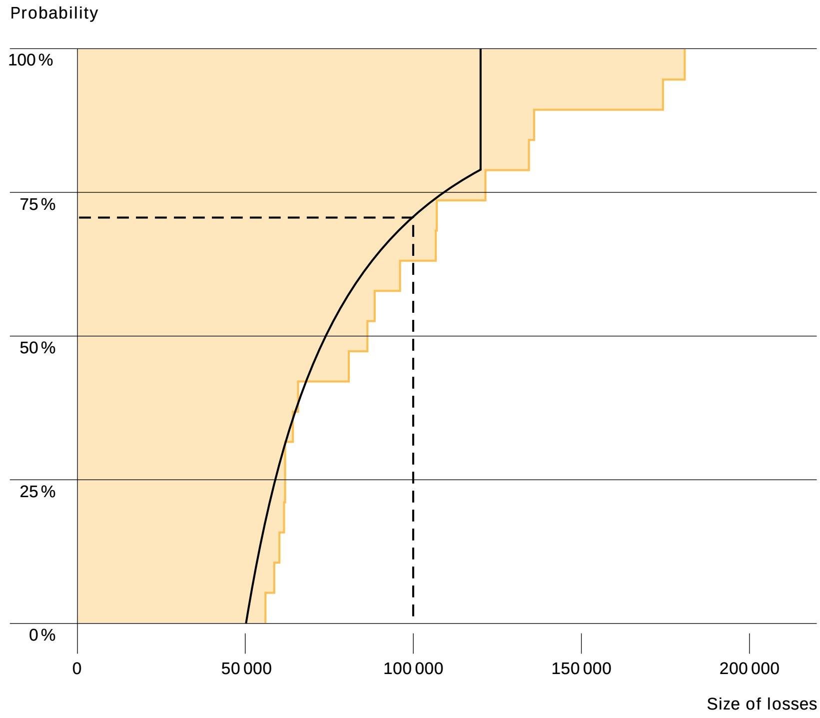
Figure 7 is obtained from figure 3 by cutting off from every loss that part in excess of 120 000; figure 8 is obtained from figure 6 in the same way.

Figure 7



The modified curve behaves like the unmodified one up to the point of truncation. Cutting off the point exceeding 120 000 will clearly not affect, for example, the probability that a loss is less than or equal to 100 000. However, at the point of truncation, the truncated curve jumps up to the value 1 (or 100%), as the probability that a loss is less than or equal to the truncation value is 1. No truncated loss can be greater.

Figure 8



7 Frequency extrapolations

The frequency is the average number of losses per year. If the frequency at a particular point, that is at the so-called low deductible, is known then it is possible to estimate the unknown frequency of losses exceeding any given high deductible. The Pareto distributions plotted in appendix 5 can be used for this purpose.

Frequency estimations for catastrophes such as earthquakes, wind storms and floods are often obtained using sources other than loss experience for the portfolio to be reinsured. For example, market statistics on losses show that for a given area a severe storm occurring approximately once every 20 years would be expected to destroy at least 2‰ of the aggregate insured values. Armed with this frequency of 1/20 storm damage in excess of 2‰ ("low loss deductible"), it is possible to estimate the frequency of storm damage in excess of 10‰ ("high loss deductible") using the procedure described here.

Example:
Let us return to industrial fire insurance, for which experience tells us that 1.6 is a reasonable value for the Pareto parameter. The frequency of losses which exceed 80 000 is equal to 2.5. How many losses per year would we expect to exceed 400 000?

This problem can be solved in three steps:

Step 1:

Calculate the ratio $\frac{\text{high deductible}}{\text{low deductible}}$.

This ratio is $\frac{400\,000}{80\,000} = 5$.[5]

Step 2:
Mark the point representing this ratio on the horizontal axis for the curves described in appendix 5. A vertical line drawn from this point will intersect the curve corresponding to $\alpha = 1.6$ at the height of 0.92. This height is the probability that a loss greater than the low deductible will attain at most the high deductible. We want to find the complementary probability, ie the probability that a loss will exceed this value. This probability is
1 – 0.92 = 0.08.

Step 3:

Frequency at high deductible	=	Frequency at low deductible	×	Probability that high deductible will be exceeded

The frequency of losses exceeding 400 000 is therefore
$2.5 \times 0.08 = 0.20$

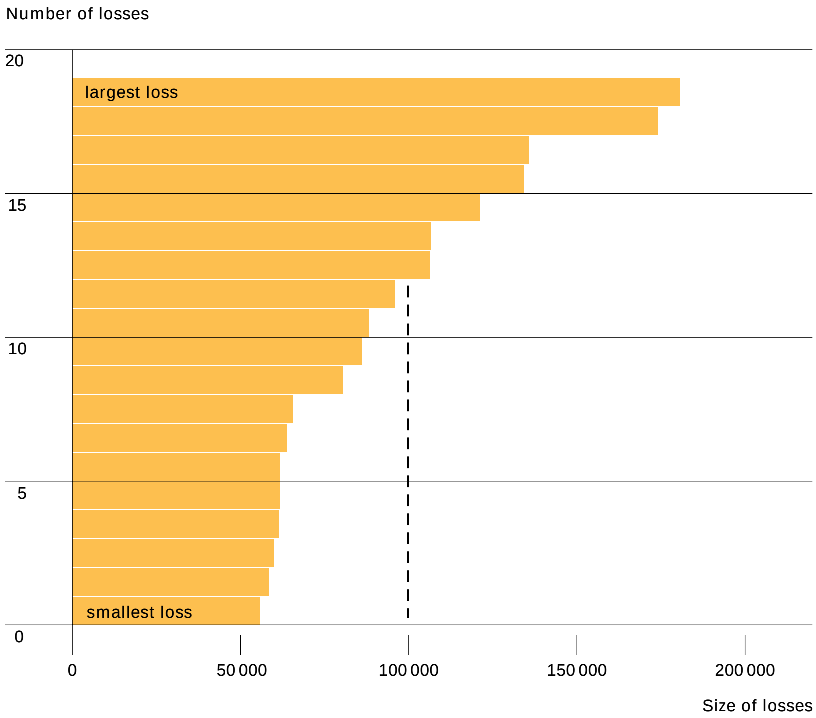
The curves plotted in appendix 5 illustrate the influence of the Pareto parameter upon loss frequency:

The lower the Pareto parameter, the higher the frequency of large losses in relation to the frequency of small losses.

[5] We would be understandably reluctant to use a high loss area with limited experience as a basis from which to extrapolate a low loss area for which there should be more information available. For this reason, loss frequencies are usually extrapolated "bottom up", ie the point of estimation (the "high deductible") is higher than the point of calibration (the "low deductible"). However, it does not need to be substantially higher. Any value greater than 1 may be used for our ratio. Still, it is mathematically possible to calculate the frequency of lower losses from known greater losses. This procedure may be applied if the underwriter has already set a minimum price for the top layer. Using the price for this top layer, he can then calculate consistent prices for lower layers (see section11).

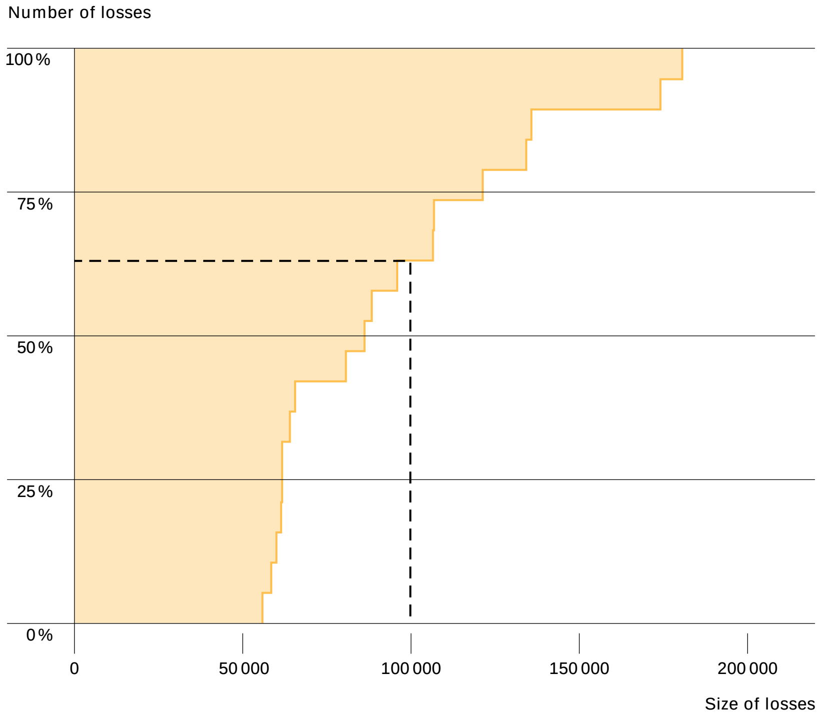
Let us look at figure 3:

Figure 3



The total of all observed losses is obviously represented by the area they cover in the graph. The average loss is therefore equal to this same area divided by the total number of losses, ie by 19. If we draw a graph similar to figure 3, but choose 1/19 (or 5.26%) instead of 1 as the height of each loss, then the area covered by all losses would represent the average loss. This again gives us figure 4.

Figure 4



The average loss is equal to the area above the flight of stairs, delimited by the vertical axis and the horizontals at the heights 0% and 100% (or 1). The same is true of truncated distributions (see section 6). It is also valid for the Pareto distribution (unlimited and truncated). There is only a difference in terminology:
In mathematical models like the Pareto distribution, we do not speak of average but of expected values. Accordingly, the area above the smooth curve in figure 8 corresponds to the expected truncated loss.

Figure 8

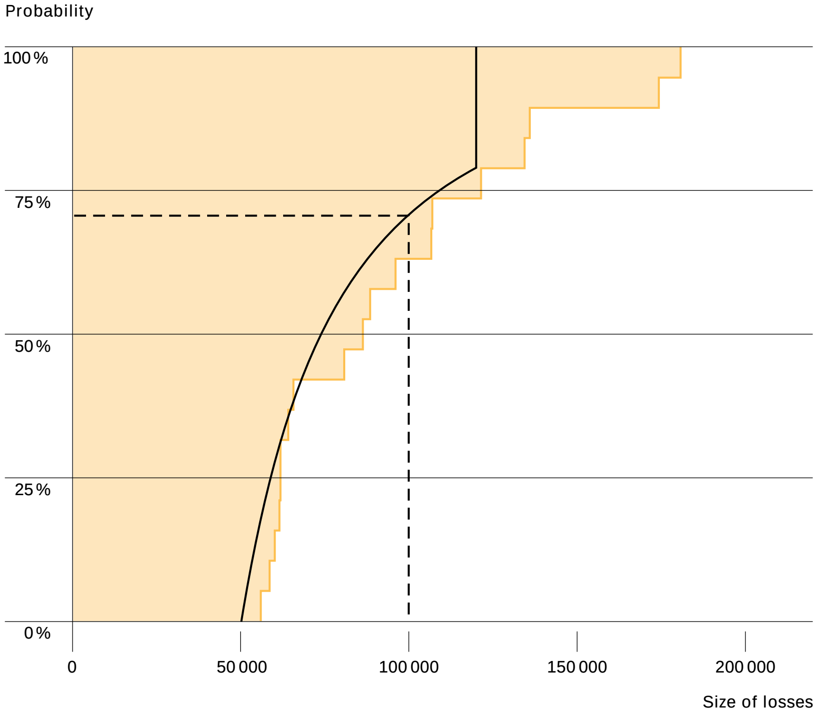


Figure 8 shows how the expected loss depends on the shape of the curve, that is, the flatter the curve, the greater the expected loss. We know from section 6 and appendix 5 that the curves defined by small Pareto parameters are flatter than those defined by large Pareto parameters. We can thus describe this relation in the following way:

The smaller the Pareto parameter, the larger the expected loss.

Figure 9 is a modification of figure 7, showing the truncated excess losses instead of the losses from ground.

Figure 9

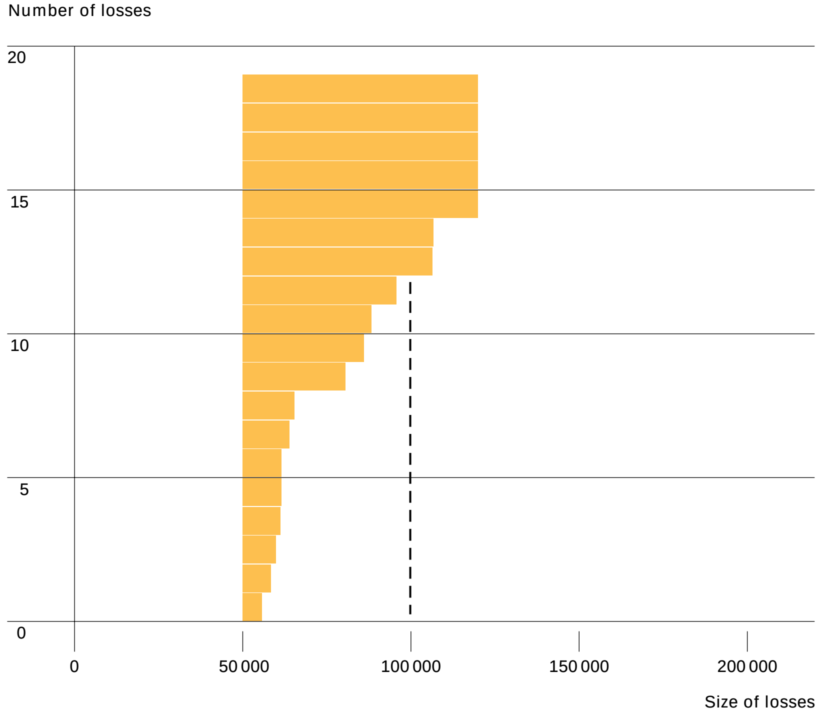
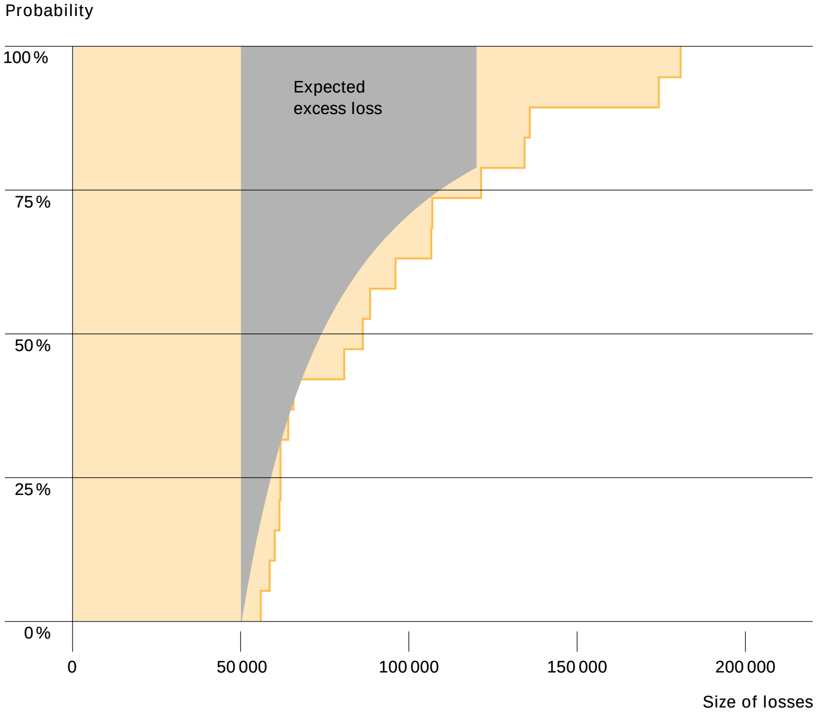


Figure 10 is drawn from figure 8 in the same manner. The expected excess loss is the expected loss less the deductible and this is represented in figure 10 by the area delimited by the curve, the verticals at 50 000 and 120 000 and the horizontal at 100%.

Figure 10



Given the Pareto parameter, the deductible and the cover, the expected excess loss can be calculated. The formulas for this calculation can be found in appendix 1. Yet in practice these are not needed, since we can also determine expected excess losses with the help of the curves in appendix 6.[6] They represent

$$\frac{\text{expected excess loss}}{\text{cover}}$$

as a function of the ratio

$$\frac{\text{deductible + cover}}{\text{deductible}}$$

(also known as the relative length of the layer).

[6] Pareto distributions: expected excess loss

To do so, we need to know the Pareto parameter α, the deductible and the cover. We can then establish the expected excess loss using the following three steps.

Example:

α	=	1.4
deductible	=	80 000
cover	=	160 000

Step 1:

Calculate the ratio $\dfrac{\text{deductible + cover}}{\text{deductible}}$.

This ratio is $\dfrac{240\,000}{80\,000} = 3.$

Step 2:
Mark the point representing this ratio on the horizontal axis of the curves described in appendix 6. The vertical from this point intersects the curve corresponding to α = 1.4 at the height 0.44. The average excess loss should therefore amount to 44% of the cover.

Step 3:
Expected excess loss = curve value $\times$ cover

The expected excess loss is therefore 0.44 $\times$ 160 000 = 70 400.

As a second illustration, take the layer 60 000 xs 60 000 and the Pareto parameter α = 1.77 and check the expected excess loss against the actual average excess loss determined from tables 1, 2 and 3.

Step 1:

$$\frac{\text{deductible + cover}}{\text{deductible}} = 2$$

Step 2:
curve value = 0.54

Step 3:
expected excess loss = 0.54 $\times$ 60 000 = 32 400.

The average excess loss was 32 301, surprisingly near the value the Pareto model would have predicted.

9 The expected excess loss burden

Risk premiums are usually calculated using the following equation:

Risk premium = expected frequency $\times$ expected loss.

Having learnt how to extrapolate frequencies (section 7) and to determine expected excess losses (section 8), we can now calculate the expected excess loss burden, or in other words the risk premium, for an excess of loss treaty. This relation is simply:

expected excess loss burden = expected frequency $\times$ expected excess loss

Example:
Let us assume a Pareto parameter α of 1.5 and a frequency at the deductible 100 000 of 4.5. What is the expected excess loss burden of the layer 500 000 xs 500 000?

First we determine the expected frequency at the deductible 500 000 as detailed in section 7.

Step 1:

$$\frac{\text{high deductible}}{\text{low deductible}} = \frac{500\,000}{100\,000} = 5$$

Step 2:
The probability that a loss will reach at most 500 000 is 0.91 (appendix 5). Therefore the probability that a loss will exceed 500 000 is 1–0.91 = 0.09

Step 3:
expected frequency at 500 000 = 4.5 $\times$ 0.09 = 0.405

Secondly, we determine the expected excess loss according to section 8.

Step 1:

$$\frac{\text{deductible + cover}}{\text{deductible}} = \frac{1\,000\,000}{500\,000} = 2$$

Step 2:
curve value (appendix 6) = 0.59

Step 3:
expected excess loss = 0.59 $\times$ 500 000 = 295 000.

Finally, the expected excess loss burden, ie the risk premium, is:

expected frequency $\times$ expected excess loss = 0.405 $\times$ 295 000 = 119 475.

We now return to our data in section 3. From tables 1 to 3, we can calculate the Pareto parameter using the method described in appendix 2. It proves to be 1.77, a figure which seems reasonable if we consider that Pareto parameters for industrial fire portfolios are usually between 1.5 and 2.5.

What price would we charge for a cover of 100 000 xs 100 000?

Note that this cover would have been hit several times during the first three years, but without a total loss.

For further calculations we require the GNPIs* (see table 5).

Table 5

Year	Index	GNPI	GNPI indexed to 119.0	Number of losses exceeding 50 000
1	110.6	4 630 000	4 981 646	8
2	113.2	5 110 000	5 371 820	5
3	117.4	5 690 000	5 767 547	6
1–3			16 121 013	19
4	119.0	6 250 000	6 250 000	?

* GNPI is the abbreviation for Gross Net Premium Income. It designates that premium which remains for the direct insurer from a portfolio before deduction of acquisition and management costs (gross) but following deduction of all premiums for proportional reinsurance (net).

Frequency estimation:

First we estimate the frequency at a deductible of 50 000. Assuming that the portfolio is unchanged, the best estimator is:

$$\text{frequency} = \frac{\text{total number of past losses}}{\text{total corresponding GNPI}} \times \text{new GNPI}$$

$$= \frac{19}{16\,121\,013} \times 6\,250\,000$$

$$= 7.37$$

Secondly, we extrapolate this frequency to the frequency at the deductible 100 000 according to section 7.

Step 1:

$$\frac{\text{high deductible}}{\text{low deductible}} = \frac{100\,000}{50\,000} = 2$$

Step 2:
The curve value from the distribution function (appendix 5) at the horizontal 2 is 0.71. The probability that a loss will exceed 100 000 is:
1–0.71 = 0.29.

Step 3:
frequency at 100 000 = frequency at 50 000 × probability of exceeding 100 000
= 7.37 × 0.29
= 2.14

Expected excess loss:

We calculate the expected excess loss according to section 8.

Step 1:

$$\frac{\text{deductible + cover}}{\text{deductible}} = 2$$

Step 2:
The curve value for the expected excess loss (appendix 6) at the horizontal 2 is 0.54.

Step 3:
expected excess loss = curve value × cover
= 0.54 × 100 000
= 54 000

Expected excess loss burden:

According to section 9, it is

expected excess loss burden = expected frequency × expected excess loss
= 2.14 × 54 000
= 115 560

Table 4 shows that the actual excess loss burden in year 4 was 125 600. This amounts to around 10 000 more than the excess loss burden to be expected on such a treaty over a period of many years, presuming of course that our model's assumptions are correct. Since individual losses cannot be predicted, these two results are unlikely ever to match exactly. The actual loss burden will differ from the expected loss burden to a greater or lesser degree. Such fluctuation creates the need for insurance and reinsurance.

The procedures detailed above enable the underwriter to estimate the risk premium for any layer within an excess of loss programme. Once the loss frequency for a given deductible is known, it is possible to calculate the loss frequency for any higher deductible. Then, using the expected excess loss, we can calculate the risk premium.

However, in practice it is common that once the Pareto model parameters and the risk premium for a single reference layer have been established, the risk premium for any other given layer is set automatically. In other words, once the risk premium for one layer of an excess of loss programme is known, it becomes possible to extrapolate risk premiums directly for additional layers, without needing to establish frequency and expected excess loss separately. When applying the formula for this calculation, we need to specify whether the Pareto parameter is $\alpha = 1$ or any other positive value.

$$RP_1 = \frac{\ln RL_1}{\ln RL_0} \cdot RP_0 \quad (\text{if } \alpha = 1)$$

$$RP_1 = \frac{\frac{DE_1}{DE_0}}{\left[\frac{DE_1}{DE_0}\right]^{\alpha}} \cdot \frac{\left[RL_1^{[1-\alpha]} - 1\right]}{\left[RL_0^{[1-\alpha]} - 1\right]} \cdot RP_0 \quad (\text{if } \alpha \neq 1)$$

(A list of symbols and abbreviations can be found in appendix 3.)

Let us use the example from section 9 as an illustration and assume that $\alpha = 1.5$ and that the frequency of losses over 100 000 is 4.5. In order to calculate the risk premium for a 100 000 xs 100 000 cover, we use the following formula:

risk premium = expected frequency × expected excess loss

Frequency extrapolation for a deductible of 100 000 is unnecessary, for it is already known to be 4.5. We can therefore begin to calculate the expected excess loss immediately (according to section 8).

Step 1:

$$\frac{\text{deductible + cover}}{\text{deductible}} = \frac{100\,000 + 100\,000}{100\,000} = 2$$

Step 2:
curve value (appendix 6) = 0.59

Step 3:
expected excess loss = curve value × cover
= 0.59 × 100 000 = 59 000

Using the expected frequency and the expected excess loss, we can calculate the

risk premium = 4.5 × 59 000 = 265 500.

In section 9 we used the standard procedure (frequency extrapolation, expected excess loss) to calculate the risk premium for 500 000 xs 500 000 cover as 119 475. Let us now apply the extrapolation formula:

$$\frac{DE_1}{DE_0} = \frac{\text{deductible}_1}{\text{deductible}_0} = \frac{500\,000}{100\,000} = 5$$

$$\left[\frac{DE_1}{DE_0}\right]^{\alpha} = \left[\frac{500\,000}{100\,000}\right]^{1.5} = 5^{1.5} = 11.1803$$

$$RL_1 = \frac{\text{deductible}_1 + \text{cover}_1}{\text{deductible}_1} = \frac{500\,000 + 500\,000}{500\,000} = 2$$

$$RL_0 = \frac{\text{deductible}_0 + \text{cover}_0}{\text{deductible}_0} = \frac{100\,000 + 100\,000}{100\,000} = 2$$

$$RP_1 = \text{risk premium}_1 = \frac{\frac{DE_1}{DE_0}}{\left[\frac{DE_1}{DE_0}\right]^{\alpha}} \cdot \frac{\left[RL_1^{(1-\alpha)} - 1\right]}{\left[RL_0^{(1-\alpha)} - 1\right]} \cdot \text{risk premium}_0$$

$$= \frac{5}{11.1803} \cdot \frac{\left[2^{-0.5} - 1\right]}{\left[2^{-0.5} - 1\right]} \cdot 265\,500 = 118\,735$$

The slight difference between the two results (119 475 as opposed to 118 735) is due to imprecise reading of the plotted curves.[7]

[7] On first sight, the extrapolation formulas seem much more complicated than the graphical method of calculation described earlier. Yet if these formulas are stored in a calculator, the risk premium for any given layer can be obtained at the touch of a button. All procedures contained in this brochure can be expressed as formulas. For further details of formulas, please refer to the Swiss Re publication entitled "The Pareto model in property reinsurance: Formulas and applications".

This is common practice for the underwriter, for example if the primary insurer changes the excess of loss limits at short notice during treaty negotiation.

In section 7 we mentioned that frequency extrapolation always takes place "bottom up", ie from a low to a high deductible. However it is common for the underwriter to decide not to make his capacity available for less than a certain minimum price. For example, the ratio "total reinsurance price (see section 12.1) to cover", also known as the "rate on line", should not be less than 2%. If he calculates a lower price for higher layers then he may decide to set the reinsurance price for the top layer arbitrarily at his minimum price. Using this market price for the top layer, he is subsequently able to establish the price of reinsurance for the lower layers using the extrapolation formulas. By setting the Pareto parameters at a suitable (not too low!) value, he can ensure that prices for lower layers calculated in this fashion do not fall below the technical minimum reinsurance prices. Such a procedure guarantees consistent pricing and is mathematically correct.

12 Additional applications of the Pareto model

12.1 The price of reinsurance

In the previous sections, we discussed a method which allowed us to estimate the long-term average amount a reinsurer is required to pay an insurer under an excess of loss treaty. The technical term for this amount is the risk premium. However, such payouts are random, as some years will be free of any losses, whilst in others losses amounting to several times the actual risk premium may accumulate. One of the fundamental aspects of reinsurance is to smooth these fluctuations among claims incurred. To do this, the reinsurer needs capital funds, and these are subject to interest payments. The interest payable on these capital funds, the administrative costs and the risk premium together form, in essence, the reinsurance price.

The reinsurance price comprises the following elements:

risk premium (= expected long-term average loss burden)
\+ uncertainty loading
(= loading for uncertain risk premium estimates[8])
\+ fluctuation loading
(= loading for taking over fluctuations in the loss process)
\+ expense loading
(= loading for managerial expenses and external costs such as taxes or brokerage)

Of these four elements of the overall reinsurance price, accurate estimation of the risk premium is the most important. Although a reinsurer's estimate of the risk premium may be correct on average, it will nevertheless be too high in some cases and too low in others. Primary insurers usually select a reinsurer on the basis of price, with the consequence that reinsurers tend to obtain treaty business where they have underestimated the risk premium. Total risk premiums in the reinsurer's portfolio are thus likely to be too low. This may be compensated by including an uncertainty loading in the reinsurance price. This is usually set at around 10% of the risk premium for the total portfolio. Risk premium estimations become less certain as the rate on line decreases and it is therefore advisable to raise the uncertainty loading for treaties with a low rate on line.

[8] You will notice that the terms “uncertainty loading” and “fluctuation loading” are often used synonymously in technical literature. However we want to make a conscious differentiation between the fluctuation risk and the risk of inaccurate premium estimation. The charge for the latter is defined therefore as the “uncertainty loading”.

The expense loading is mathematically easy to calculate, but the practical difficulties of internal cost allocation should not be underestimated. In addition to percentage loadings such as brokerage and taxes, there may also be a substantial sum payable for internal administrative expenses, depending on the cost distribution formula selected.

In contrast, the fluctuation loading poses a more demanding mathematical problem, depending on the model selected. We will concern ourselves only with a method of calculation which uses the Pareto model. The size of the fluctuation loading for a given treaty depends essentially on the potential which the treaty has of unbalancing the reinsurer's entire portfolio, on the size of his free reserves and on his willingness to put them at stake.[9] Readiness to accept risks is a managerial decision. Once it is taken, it should be applied consistently to all treaties in the reinsurer's portfolio. Focusing on that fluctuation ceded by the primary insurer to the reinsurer (and not on the additional fluctuation taken on by the reinsurer!) will lead to a fluctuation loading which is proportional to the variance (measure for fluctuation potential) of the excess loss burden.

12.2 Variance – Chebyshev's inequality

Section 9 shows how to calculate the expected excess loss burden. It is the value which the average annual excess loss burden would tend to after a very long period. The actual excess loss burden in any particular year will deviate from its expected value. The variance is the expected value of the square of such deviations. The more the annual excess loss burden is likely to deviate from its expected value (the risk premium), the greater the variance will be.

Chebyshev's inequality, which is proven in any textbook on probability theory, shows the significance of variance as a measure of susceptibility to fluctuation. It states:

The probability that a deviation will exceed a particular value is no greater than the ratio of the variance to the square of this particular value.

[9] Probability theory measures this willingness by the so-called "probability of ruin".

Let us illustrate this by an example: in section 9 we calculated the expected excess loss burden of a layer 500 000 xs 500 000 as 119 475 (using a Pareto parameter $\alpha = 1.5$). The curves in appendix 7 help to calculate the corresponding variance and indicate the ratio

$$\frac{\text{variance}}{\text{cover} \times \text{expected excess loss burden}}$$

as a function of the ratio

$$\frac{\text{deductible + cover}}{\text{deductible}}.$$

Step 1:
Calculate the ratio

$$\frac{\text{deductible + cover}}{\text{deductible}}$$

$$\frac{500\,000 + 500\,000}{500\,000} = 2$$

Step 2:
Look up the curve value (appendix 7) corresponding to the horizontal value as calculated in step 1. A vertical line throught this point would intersect the curve defined by the Pareto parameter $\alpha = 1.5$ at a value of 0.83.

Step 3:

$$\begin{aligned}\text{variance} &= \text{curve value} \times \text{cover} \times \text{expected excess loss burden}\\ &= 0.83 \times 500\,000 \times 119\,475\\ &= 49\,582\,125\,000\end{aligned}$$

What is the probability that the following year's excess loss burden will exceed 500 000? Chebyshev's inequality shows that this probability is no bigger (but probably much smaller) than

$$\frac{\text{variance}}{\text{deviation}^2} = \frac{49\,582\,125\,000}{(500\,000 - 119\,475)^2} = 0.34 \text{ or } 34\%.$$

12.3 Determining the fluctuation loading

Like an extrapolated frequency or an expected excess loss, the fluctuation loading is determined in three steps. For an example, we revert to section 9, ie:

Pareto parameter α	1.5
deductible	500 000
cover	500 000
expected frequency	0.405

Step 1:

Calculate the ratio $\dfrac{\text{deductible + cover}}{\text{deductible}}$.

For example:

$$\frac{1\,000\,000}{500\,000} = 2$$

Step 2:
Look up the curve value corresponding to 2 in appendix 5 (Pareto parameter α = 1.5): it is 0.83

Step 3:
fluctuation loading = curve value (appendix 7)

$$\times \frac{\text{cover}^{10}}{1000}$$

$$\times \frac{\text{expected excess loss burden}^{10}}{1000}$$

× currency exchange rate[11]

× accepted share of the treaty

× fluctuation factor

[10] expressed in original currency, ie that of the primary insurer.

[11] The primary insurer's monetary unit, expressed in units of the reinsurer's currency.

The third step calls for some supplementary comments:

1. The variance is

   curve value × cover × expected excess loss burden.

   Usually these multiplications result in very high numbers which are cumbersome to deal with. For this reason, it is better to divide the cover and the expected excess loss burden by 1 000 and adjust the fluctuation factor in the last line accordingly.

2. The appearance of the exchange rate must not be misinterpreted: it does not mean that the fluctuation loading is calculated in a different currency to the expected excess loss burden. They are both determined in the original (ie the primary insurer's) currency. It is nevertheless clear that a cover of GBP 500 000 is more likely to unbalance the reinsurer's portfolio than a cover of ITL 500 000. The fluctuation loading for the more risky cover must be higher than that for the less risky one. Another plausible explanation is the fact that the reinsurer operates his technical account mainly in the currency of the country where he is established. Therefore any considerations involving his portfolio as a whole are obviously directed by that currency.

3. Likewise, a share of 100% will entail a more significant commitment vis-à-vis the reinsurer's existing portfolio than a share of only 10%. Heavy excess losses, as may occur under new substantial covers, may easily unbalance the relevant portfolio, especially small ones. In such cases, the fluctuation loading must be large if a significant treaty share is to be written. This is the reason why, at the same level of fluctuation loading (in percent of the risk premium), larger reinsurers are more able to accept large shares than small companies.

4. The fluctuation factor is determined either by mathematical methods or by the procedure described in section 12.4 “Determining the fluctuation factor”. It reflects the reinsurer's readiness to assume a risk. Fluctuation factors vary from reinsurer to reinsurer, resulting in each reinsurer being prepared to accept different shares of the same treaty.

In our example, we will assume that:
– the primary insurer's currency unit is GBP, that of the reinsurer CHF. The exchange rate is 2.3.
– the reinsurer's share of the treaty is to be 40%.
– the reinsurer's fluctuation factor is 0.5.

In accordance with step 3, the fluctuation loading is:

fluctuation loading = $0.83 \times 500 \times 119.475 \times 2.3 \times 0.4 \times 0.5 = 22\,808$

12.4 Determining the fluctuation factor

The fluctuation factor should be the same for all treaties. For practical purposes it can be calculated in the following way: The account manager responsible for a particular account should state what the fluctuation loading of a typical treaty should be. He may feel that for a fire treaty of GBP 500 000 xs 500 000 with a frequency of 0.405 he would accept a share of 40%, provided that he finds the uncertainty loading and administrative expense loading satisfactory and if the fluctuation loading is 20% of the risk premium. (The reader will notice that this is the treaty used in sections 9 and 12.3 “Determining the fluctuation loading”. All important figures are calculated in those two sections.) He then calculates:

fluctuation factor = fluctuation loading

: curve value (appendix 7)

$: \dfrac{\text{cover}}{1000}$

$: \dfrac{\text{expected excess loss burden}}{1000}$

: currency exchange rate[12]

: accepted share of the treaty

Numerically, in this example this leads to:

$$\text{fluctuation factor} = \frac{119\,475 \times 0.2}{0.83 \times 500 \times 119.475 \times 2.3 \times 0.4} = 0.52$$

[12] The primary insurer's monetary unit, expressed in units of the reinsurer's currency.

12.5 Determining the share

For this model of fluctuation loading calculation, the reinsurance price and the share accepted by the reinsurer are closely linked. In practice, every excess loss offer should be accompanied by a statement of the share which the reinsurer is prepared to write at the proposed price. Should the price be altered during the course of negotiations, the reinsurer must consider whether he wishes to adjust the share which he is willing to write.

The figures required for our example are the following:

Pareto parameter	1.5
deductible	GBP 500 000
cover	GBP 500 000
frequency	0.405

Let us further assume a GNPI of 6 000 000. The treaty is offered at a rate of 2.5% GNPI. What share does the reinsurer accept?

The fluctuation loading offered is
(premium rate × GNPI) – risk premium – uncertainty loading – expense loading

In our example, the reinsurer requests an uncertainty loading of 10% of the risk premium and an expense loading of 7.5% of the risk premium.

fluctuation loading =
$0.025 \times 6\,000\,000 - 119\,475 - (0.1 \times 119\,475) - (0.075 \times 119\,475) = 9\,617$

The share is calculated as
share accepted = fluctuation loading

: curve value (appendix 7)

$: \frac{\text{cover}}{1\,000}$

$: \frac{\text{expected excess loss burden}}{1\,000}$

: currency exchange rate[13]

: fluctuation factor

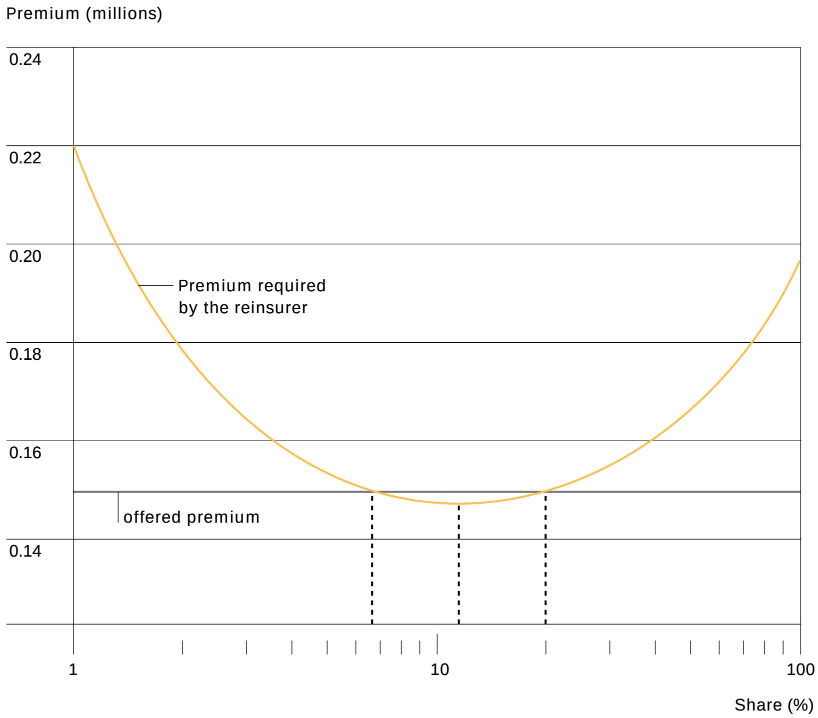
[13] The primary insurer's monetary unit, expressed in units of the reinsurer's currency.

Numerically, we get

$$\text{share accepted} = \frac{9617}{0.83 \times 500 \times 119.475 \times 2.3 \times 0.52} = 0.16 \text{ or } 16\%.$$

In practice, the reinsurer often divides his internal costs into a fixed amount and a percentage of the premium. This complicates the determination of the share. The curve describing the premium required by the reinsurer resembles a parabola. If the reinsurer accepts a very small share of the treaty, the total reinsurance price is dominated by fixed costs. If, on the other hand, he accepts a large share, then the total price is directed mainly by the fluctuation loading.

Figure 11



The left-hand intersection of the curve (= required premium) with the horizontal line (= offered premium) tells the reinsurer the minimum share required to achieve his target returns. At the lowest level of the curve, he obtains the highest possible percentage fluctuation loading, relative to the risk premium. At the right-hand intersection, he obtains the largest possible fluctuation loading in absolute terms to meet his needs.

Appendices

Appendix 1
Mathematical principles

The Pareto distribution function is defined by

$$F(x) = 1 - \left[\frac{OP}{x}\right]^{\alpha} \text{ for } \alpha > 0 \text{ and } 0 < OP \leq x$$

If $y = \left[\frac{x}{OP}\right]$ then this distribution function becomes

$$G(y) = 1 - \frac{1}{y^{\alpha}}$$

Appendix 5 shows graphs of this function for different Pareto parameters α.

The truncated distribution is given by

$$F(x) = \begin{cases} 1 - \left[\frac{OP}{x}\right]^{\alpha} & \text{if } OP \leq x < EP \\ 1 & \text{if } x \geq EP \end{cases}$$

The representation of expected values as the area above a distribution curve can be shown by integration by parts.

If X is the loss variable, then X–OP is the excess loss variable. The expected values E[X–OP] and E[(X–OP)^2] for truncated distributions are listed in table 6.

Table 6

α	$E[X-OP]$	$E[(X-OP)^2]$
1	$OP \cdot \ln\left[\frac{EP}{OP}\right]$	$2 \cdot OP^2 \cdot \left[\frac{EP}{OP} - 1 - \ln\left[\frac{EP}{OP}\right]\right]$
2	$OP - \frac{OP^2}{EP}$	$2 \cdot OP^2 \cdot \left[\frac{OP}{EP} - 1 + \ln\left[\frac{EP}{OP}\right]\right]$
$\neq 1, \neq 2$	$\frac{OP - EP \cdot \left[\frac{OP}{EP}\right]^{\alpha}}{\alpha - 1}$	$2 \cdot OP^2 \cdot \left(\frac{\left[\frac{EP}{OP}\right]^{1-\alpha} - 1}{\alpha - 1} - \frac{\left[\frac{EP}{OP}\right]^{2-\alpha} - 1}{\alpha - 2}\right)$

Appendix 6 shows graphs of $\frac{E[X-OP]}{EP-OP}$ as a function of $\frac{EP}{OP}$

for different Pareto parameters α.

Let N denote the number of losses exceeding OP per time unit. Then the total excess loss burden per time unit is

$$Z = [X_1 - OP] + [X_2 - OP] + \dots\dots\dots\dots + [X_N - OP].$$

Risk theory shows that the following relations hold if N and X are independent:

$$E[Z] = E[N] \cdot E[X - OP]$$

$$Var[Z] = E[N] \cdot Var[X - OP] + Var[N] \cdot E[X - OP]^2$$

If N is Poisson distributed, then $E[N] = Var[N]$ so we obtain

$$Var[Z] = E[N] \cdot E[(X - OP)^2]$$

Section 9 is based on the assumption that N and X are independent. In section 12 we further assume N to be Poisson distributed.

Appendix 7 shows graphs of $\dfrac{E[(X-OP)^2]}{[EP-OP] \cdot E[X-OP]}$

as a function of $\dfrac{EP}{OP}$ for different Pareto parameters.

Appendix 2
Estimating the Pareto parameter

Let us assume that we have available statistics of n losses each exceeding a limit OP. The Pareto parameter which best approximates the empirical distribution function defined by these statistics is obtained by:[14]

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n \cdot \ln(OP)}$$

$$= \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{x_i}{OP}\right)}$$

ln is the "natural logarithm".

Appendix 4 shows a graph of natural logarithms.

[14] The parameter value determined in this way is called "maximum likelihood estimation".

Appendix 3
Index of symbols and abbreviations

F (.)	distribution function
f (.)	density function
E (.)	expected value
Var (.)	variance
P (.)	probability

Random variables:

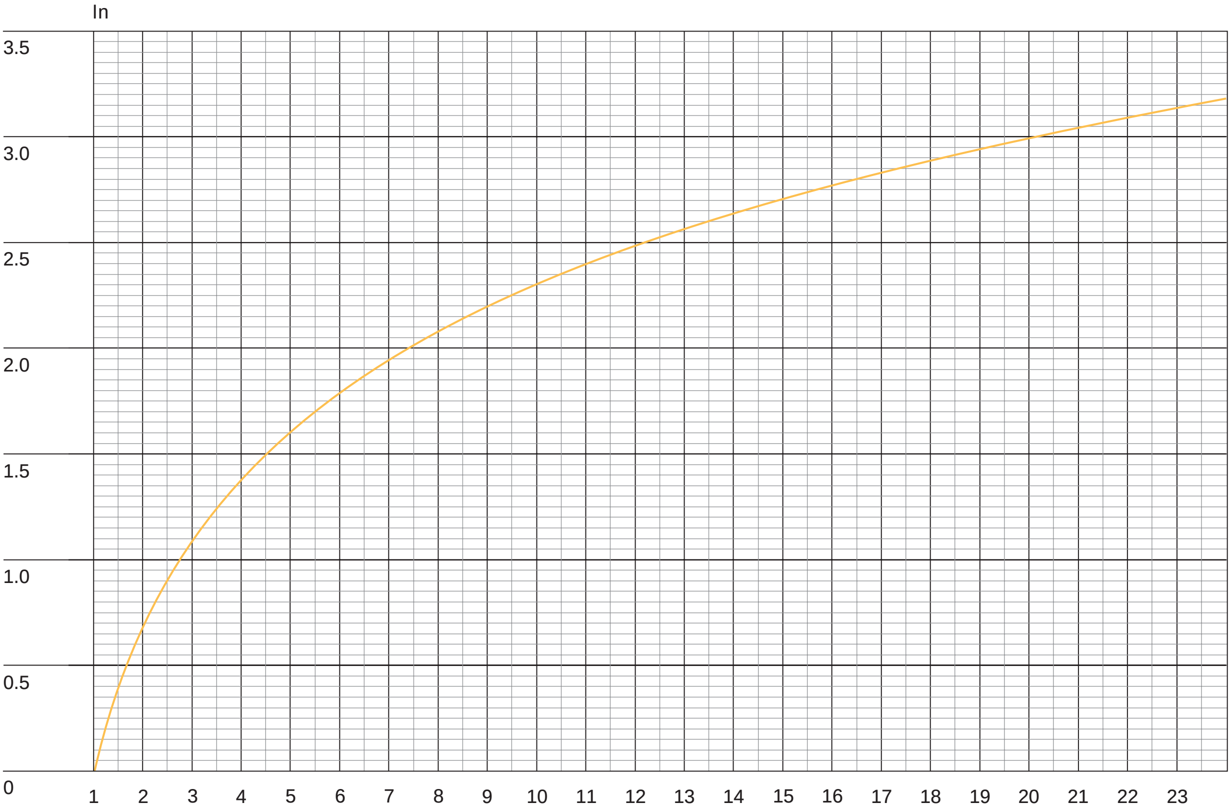
X	size of losses
N	number of losses
Z	loss burden

α	Pareto parameter
$\hat{\alpha}$	estimation of α

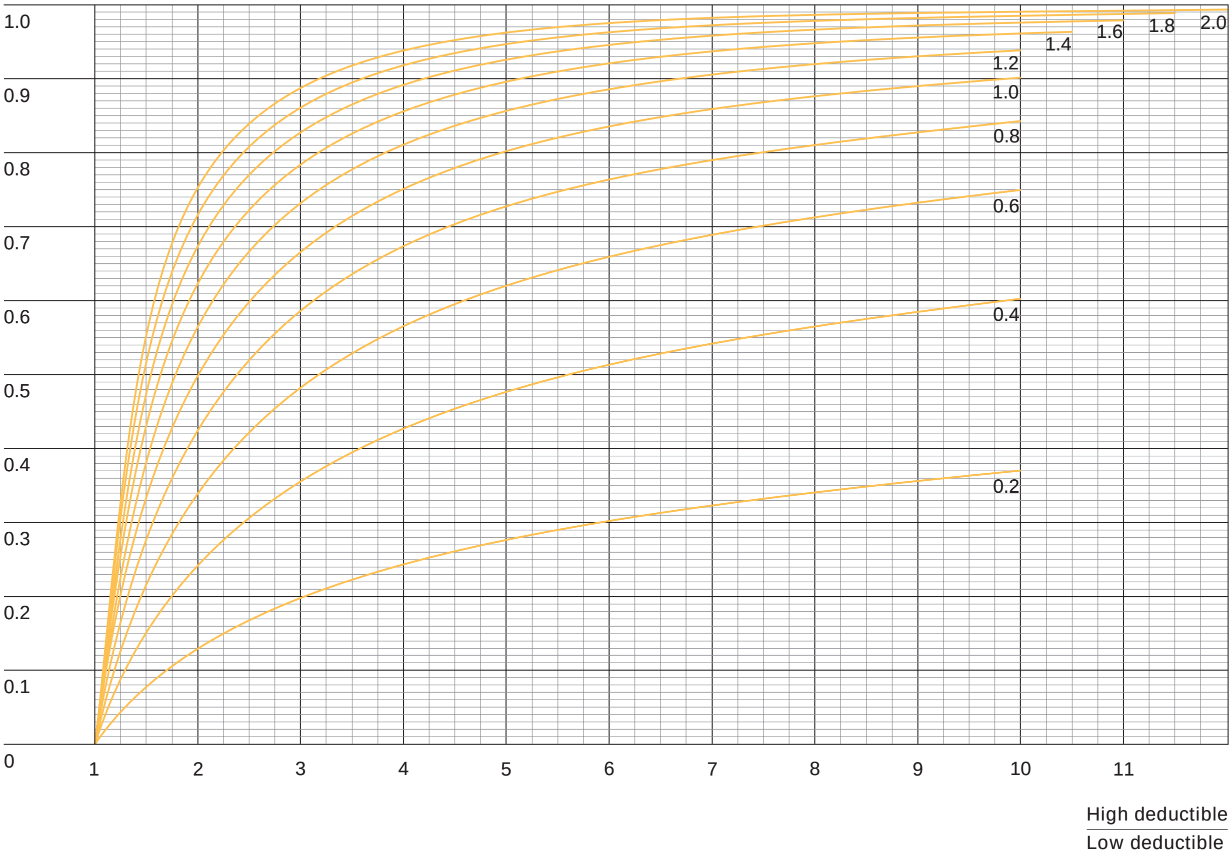
CO	cover
DE	deductible
OP	observation point
FQ	frequency
EL	expected excess loss
RP	risk premium
PR	premium

EP = DE + CO	exit point
$RL = \frac{DE + CO}{DE}$	relative length of the layer
$RROL = \frac{RP}{CO}$	risk rate on line
$ROL = \frac{PR}{CO}$	rate on line

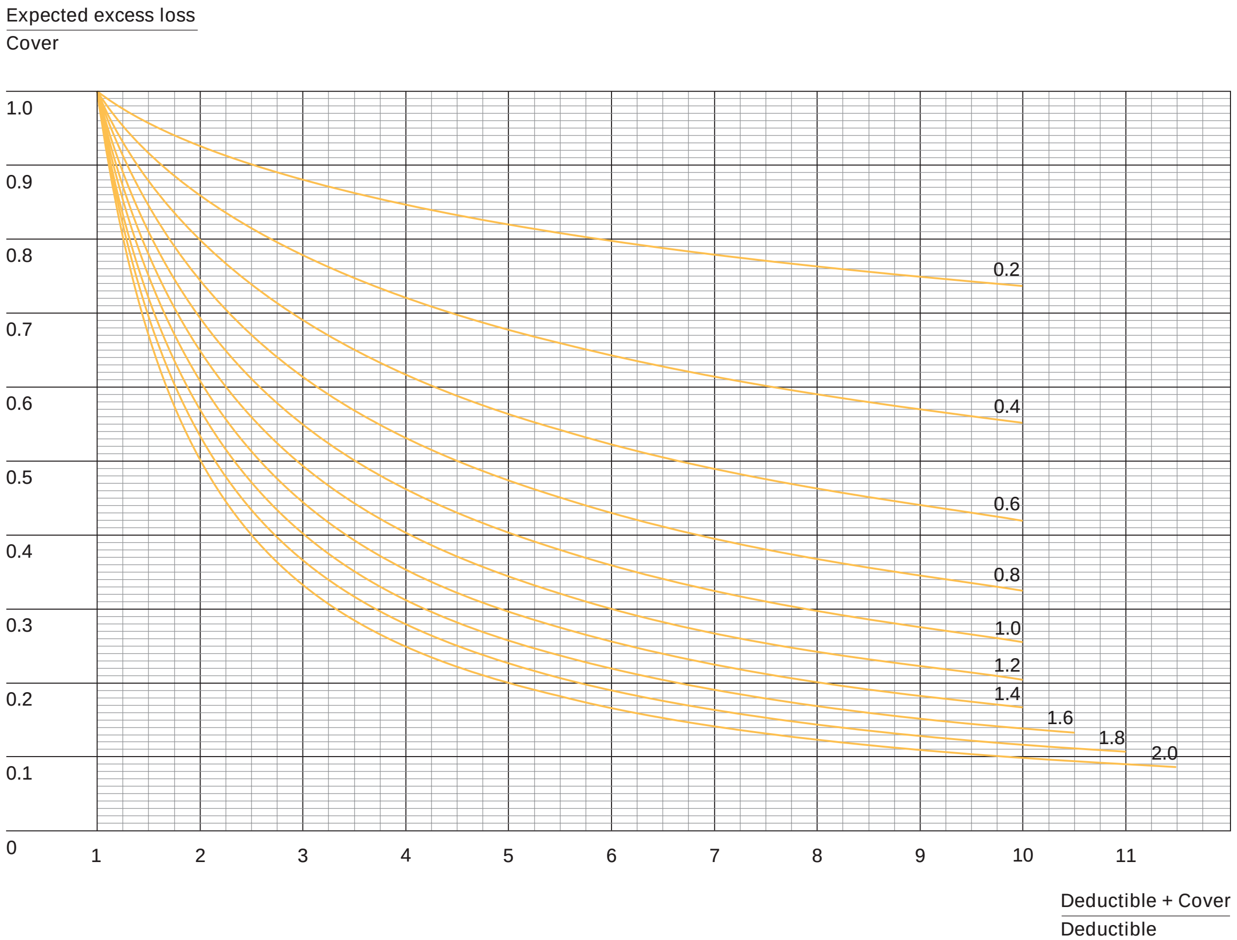
Appendix 4
Natural logarithms



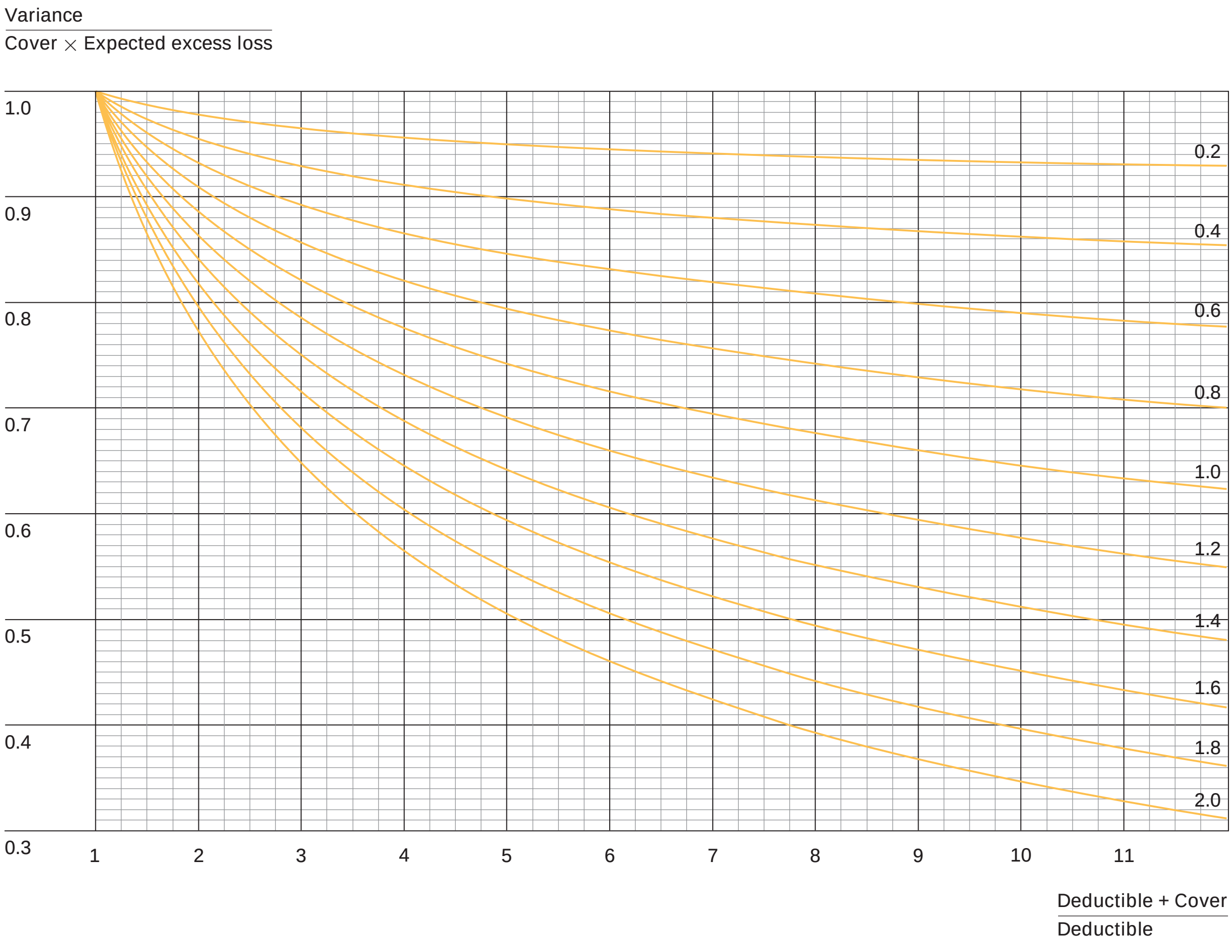
Appendix 5
Pareto distribution functions



Appendix 6
Pareto distributions: expected excess loss



Appendix 7
Pareto distributions: variance



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