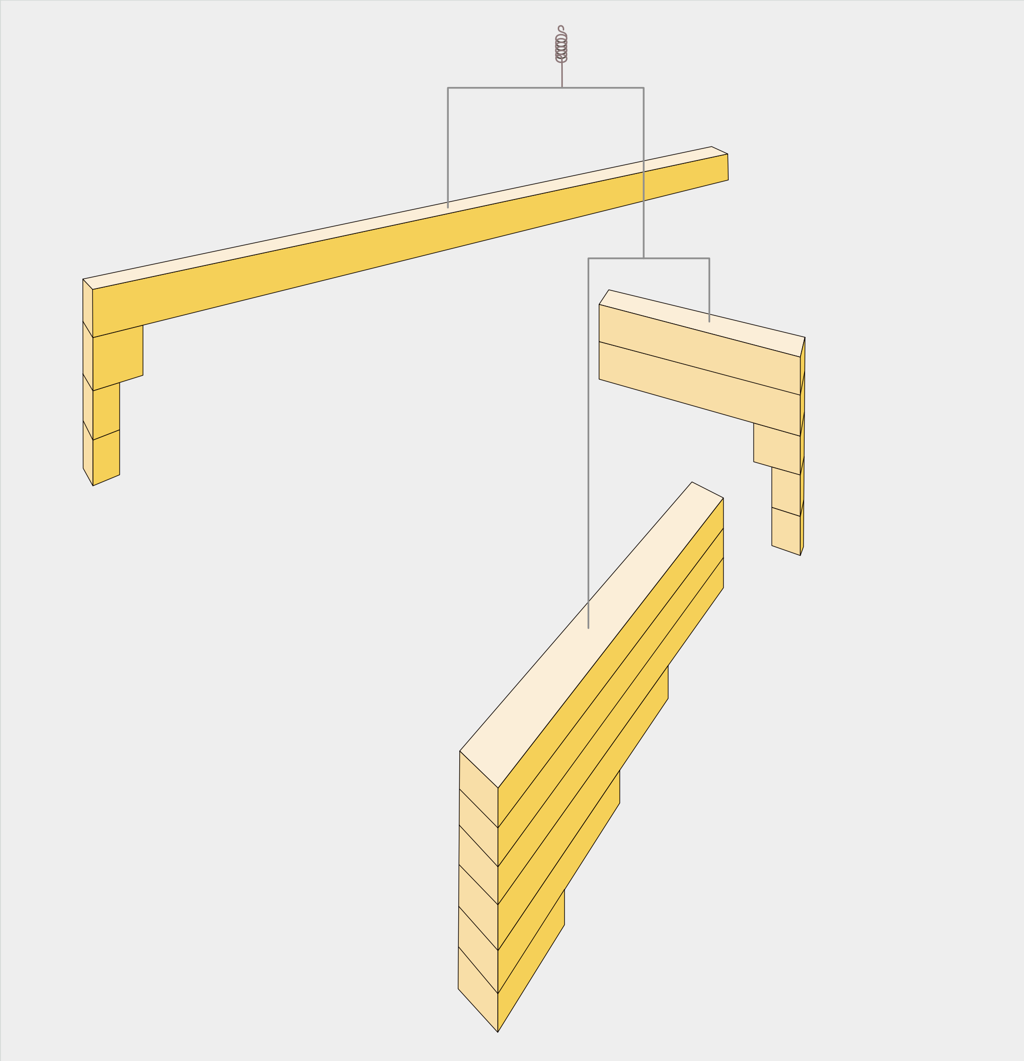
Swiss Re



Setting optimal reinsurance retentions

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Foreword

The foremost goal of reinsurance is to maintain, at an acceptable level, the coincidental fluctuations in the results of primary insurers. Author Hans Schmitter reveals how this can be effectively achieved in the following study. It is aimed equally at primary insurers, reinsurers and reinsurance brokers, who advise their clients on arranging reinsurance programmes. Yet beyond an understanding of simple algebraic formulae, no mathematical knowledge is required to understand the text.

The concepts described in this study are applicable to every class of insurance and are not limited, for instance, to the casualty or property damage classes. The publication is based on Bruno de Finetti's work on the establishment of optimal proportional retentions which was published in a 1940 article entitled "Il problema dei pieni" [1]. It also covered the fundamentals for establishing non-proportional retentions which will be elaborated upon in this publication. Because de Finetti's work remained virtually unknown outside universities, actuaries and other professionals are still to a large extent unfamiliar with practicable rules deriving from his observations. For this reason, even readers with knowledge of risk theory will also find something new in the following work.

The publication includes a compact disc with an Excel file developed by Pamela Hall which features tools to determine useable retentions. These range from simple aids to calculate numeric examples to instruments which help determine realistic distribution parameters and those tools which help in the construction of whole reinsurance programmes across several branches. The calculation methods are programmed in Visual Basic.

Thomas Hiltmann
Head of Group Product Management Casualty

1 Introduction

How can the coincidental fluctuations in the results of primary insurers be effectively maintained at an acceptable level with the help of reinsurance? This question is precisely the subject of the following study. In this context, "effectively" means at the price one is prepared to pay for reinsurance to minimise fluctuations as much as possible, not as much through price negotiation, as through the skilful use of retentions.

The meaning of retention will depend on the type of treaty. A quota share retention is the part of the business which is not reinsured, and is usually expressed as a percentage. A surplus retention is part of the sum insured or MPL (Maximum Possible Loss). It is referred to as a line and is stated as a monetary amount. Further, an excess of loss retention is the maximum amount of a loss which a primary insurer must pay – the reinsurer pays for anything over and above this. The total loss burden which the primary insurer assumes is also occasionally described as the retention. The entire set of retentions in all the reinsurance treaties of a primary insurer is usually called the reinsurance programme.

For property reinsurance, there are certain rules of thumb for setting up effective reinsurance programmes (see [4]), however, no such rules exist for the casualty branches (liability, accident, motor). Practitioners would welcome such rules because every primary insurer, in establishing its reinsurance programme, asks the following questions: which motor liability quota share reinsurance goes with a certain fire surplus reinsurance? Which own damage quota share reinsurance goes with a motor liability quota share of, for example, 30%? Should a quota share retention be protected by an excess of loss reinsurance? What does an optimally structured reinsurance programme look like? What information is required to address such concerns? Insurance companies, which use internal reinsurance to smooth out the results of their individual profit centres, ask similar questions, and these will be considered in the following study, which aims to provide solutions with both sound theoretical bases and practical viability.

2 Measuring susceptibility to fluctuations

Every risk portfolio incurs losses of greater and lesser amounts at irregular intervals. The sum of all the losses in any one year is described as the annual loss burden or simply the loss burden. The future annual loss burden is estimated on the basis of the predicted claims frequency and predicted average individual claim amount, but what actually happens usually deviates considerably from forecasts and on a purely accidental basis. A risk portfolio is considered susceptible to fluctuations when the actual loss burden is expected to deviate widely from the predicted loss burden. This somewhat vague definition of susceptibility to fluctuations can be precisely understood by consulting the following.

If, for example, in a portfolio of motor liability risks, 10,000 claims on average are expected, it would be no surprise if 100 more claims or 100 fewer occurred. Fluctuations of this order are normal when the number of claims is expected to be 10,000. However, if we consider a much smaller sample where only 20 claims are expected, then a deviation of 100 more claims would be highly unusual. A deviation of 100 less claims would, of course, be impossible. This example shows that differences between the actual and expected number of claims are more likely when the expected number of claims is larger (1,000 as opposed to 20): larger deviations occur more frequently in large samples than in small ones. Hence the expected number of claims gives the first indication of whether large or small fluctuations must be expected.

Let us now consider two portfolios, both anticipating 10,000 claims. Further, for the first portfolio each claim amounts to EUR 100 and for the second each claim amounts to EUR 10,000. In the first case one claim more or less alters the total loss burden by only EUR 100, however, in the second case it is altered by EUR 10,000. In this situation, the second portfolio is more susceptible to fluctuations than the first. When all claims cost the same amount, the loss burden composed of high claim amounts fluctuates more widely than that of low ones.

Finally let us consider another two portfolios, again expecting an equally large number of claims. In the first one, each claim amounts to EUR 1,000; the second has claims of different amounts, but the average also amounts to EUR 1,000. One would suspect the second portfolio to be more susceptible to fluctuations. However, it is necessary to dig a little deeper to justifiy this perception.

Figure 1 shows a graphical representation of all possible losses. Each claim[1] is depicted by a bar and arranged in order of size, the smallest at the bottom and the largest at the top. Such a representation is called a claim distribution. Rather than listing the number of claims from the smallest to the largest, using a scale of 0 to 100 percent of all claims on the vertical axis is helpful for making observations. The claim size described as E depicts the expected or average claim.

[1] Loss and claim have the same meaning in insurance. In actuarial science, however, the term claim is more commonly used relative to claim distribution.

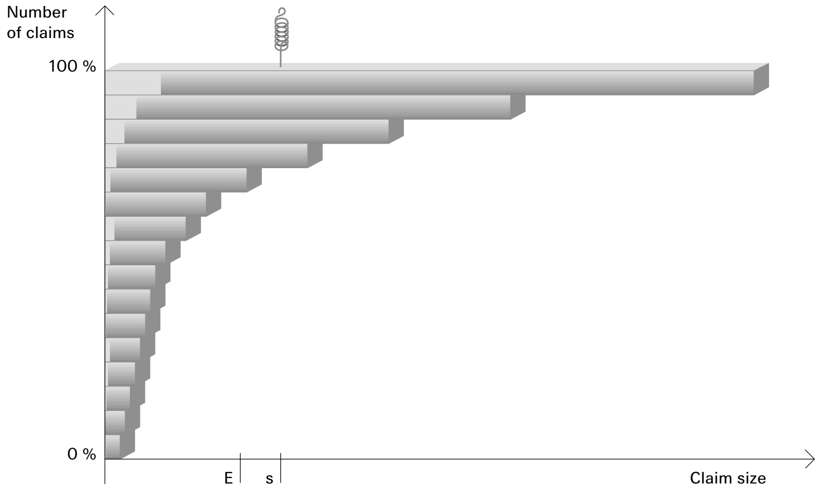

Figure 1

The whole bar formation is suspended at a distance of s from the left boundary and should be in equilibrium. The greater the difference between the individual claims, the greater s becomes. When all claims are equally high, as in Figure 2, s becomes half as big as E or 2·s is the same size as E. s cannot be smaller in relation to E than it is in this case. To measure the susceptibility to fluctuations we use the amount 2·s, ie twice the claim amount s, at which the claim distribution would be in equilibrium, together with the expected number of claims and the average individual claim size. The three figures are then multiplied together and the product is known as the variance of the loss burden:

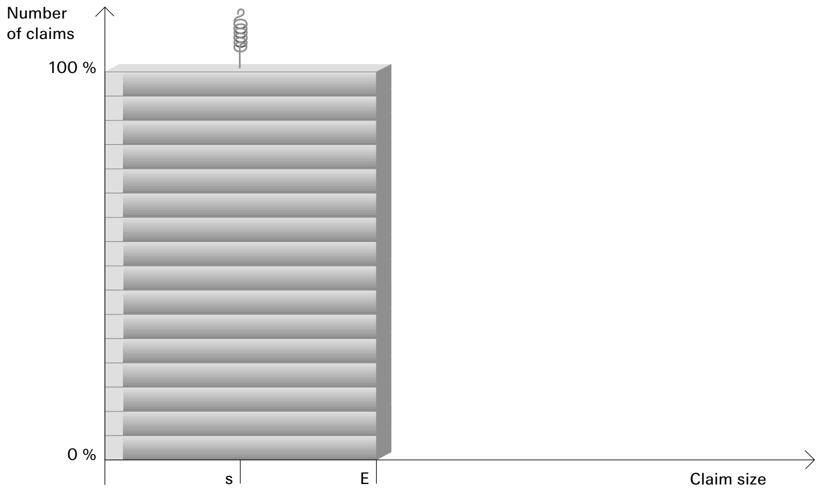

Figure 2

Variance of the loss burden = expected number of claims · expected individual claim amount·2·s

The short and clear way of writing such equations which is commonly used in non-life mathematics includes the following:

Z the loss burden,
V[Z] the variance of the loss burden,
λ the expected number of claims and
E the expected individual claim.

The equation is then as follows

(2.1) $V[Z] = \lambda \cdot E \cdot 2 \cdot s$

As we know intuitively, the variance is large when the expected number of claims is large, also when the expected individual claim amount is large and when the individual claims vary greatly from one another. A further important feature of the variance is the following:

Combining two smaller portfolios to form a larger one, the variance of the larger portfolio is equal to the sum of the two variances of the smaller portfolios, as the following numerical example illustrates. Let us consider two portfolios A and B. In A, there are on average 6 claims per year, all of the same amount of 2. In B, there is on average one claim per year at a fixed amount of 16. It is straightforward to determine the expected number of claims and the average claim of the combined portfolio. Figure 3 shows the amount of claim s at which the combined distribution would be perfectly balanced, if it could be held in suspension. Two weights are suspended from a mobile. The upper bar weight depicts a claim of 16, the lower one depicts 6 claims of 2 stacked. The two weights balance when the 16 weight hangs from an arm of length 3 and the 12 weight hangs from an arm of length 4. One deduces from Figure 3 that s can be either the sum of 1 and 4 or the difference between 8 and 3. Table 1 shows that the variance of the combined portfolio is actually the sum of the two individual partial variances (24+256 = 280).

	A	B	Combined
Expected number of claims	6	1	6+1=7
Expected loss burden	12	16	12+16=28
Expected individual claim amount	2	16	28/7=4
s	1	8	5 (Figure 3)
Variance of the loss burden	6·2·2·1=24	1·16·2·8=256	7·4·2·5=280

Table 1

Readers with knowledge of probability theory will note that in the above definition of variance the tacit assumption has been made that the number of claims is Poisson distributed.

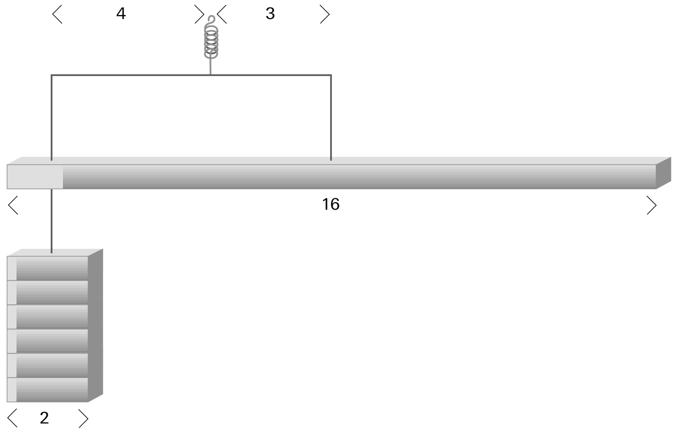

Figure 3

In Figure 4, E and 2·s are graphically depicted once again, this time, however, in the form of a rectangle with a length of 2·s, a width of E and consequently an area of E·2·s. This rectangle consists of a square with sides of length E and a smaller rectangle with sides E and 2·s-E. To multiply the two sides of the smaller rectangle together yields an area for the rectangle of E·(2·s-E). This is called the variance of the individual claim (as opposed to the variance of the annual loss burden). If we call this V, then

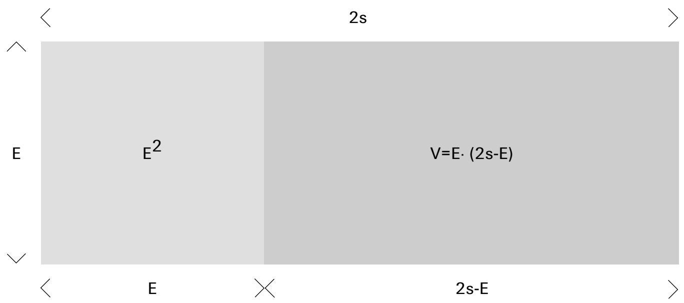

Figure 4

$$(2.2)\ E \cdot 2 \cdot s = E^2 + V$$

The variance of the loss burden can therefore also be calculated differently from that shown in equation (2.1): To replace E·2·s with E^2+V yields

$$(2.3)\ V[Z] = \lambda \cdot (E^2 + V)$$

The representation (2.3) of the variance V[Z] is more widespread than the equation shown in (2.1), in which the claim amount s appears. The claim amount s, at which a distribution would be in equilibrium, is rarely used in probability calculations and has no special name for that reason.

3 The probability of losing a large amount of capital

The variance helps estimate the probability that the loss burden will deviate from its expected value by a large amount. Chebyshev's inequality, which is proven in every textbook on probability theory, can be used for this. It says:

The probability that the deviation exceeds a certain amount is no greater than the ratio of the variance to the square of that amount.

The following numerical example illustrates these relationships. In terms of size it relates to circumstances in Swiss motor liability insurance, whereby we assume that the currency unit is the euro.

Expected number of claims:	1 000
Average individual claim:	4 000
Variance of the individual claim:	$1.02 \cdot 10^9$
Expected loss burden:	$4 \cdot 10^6$
Variance of the loss burden:	$1\,000 \cdot (4\,000 \cdot 4\,000 + 1.02 \cdot 10^9) = 1.036 \cdot 10^{12}$
Capital:	5 000 000

According to Chebyshev's inequality, the probability that the loss burden exceeds its average value of 4,000,000 by more than 5,000,000, is at the most $1.036 \cdot 10^{12} / 5{,}000{,}000^2 = 4.1\%$.

What is remarkable about this estimate is that we need to know virtually nothing about the claim distribution. It is sufficient to know the average claim and the variance.

4 Reducing the variance by reinsurance

The primary goal of reinsurance is to reduce the fluctuations of the loss burden. The variance is the measurement criterion for these fluctuations. This section illustrates how reinsurance reduces the variance of the primary insurer.

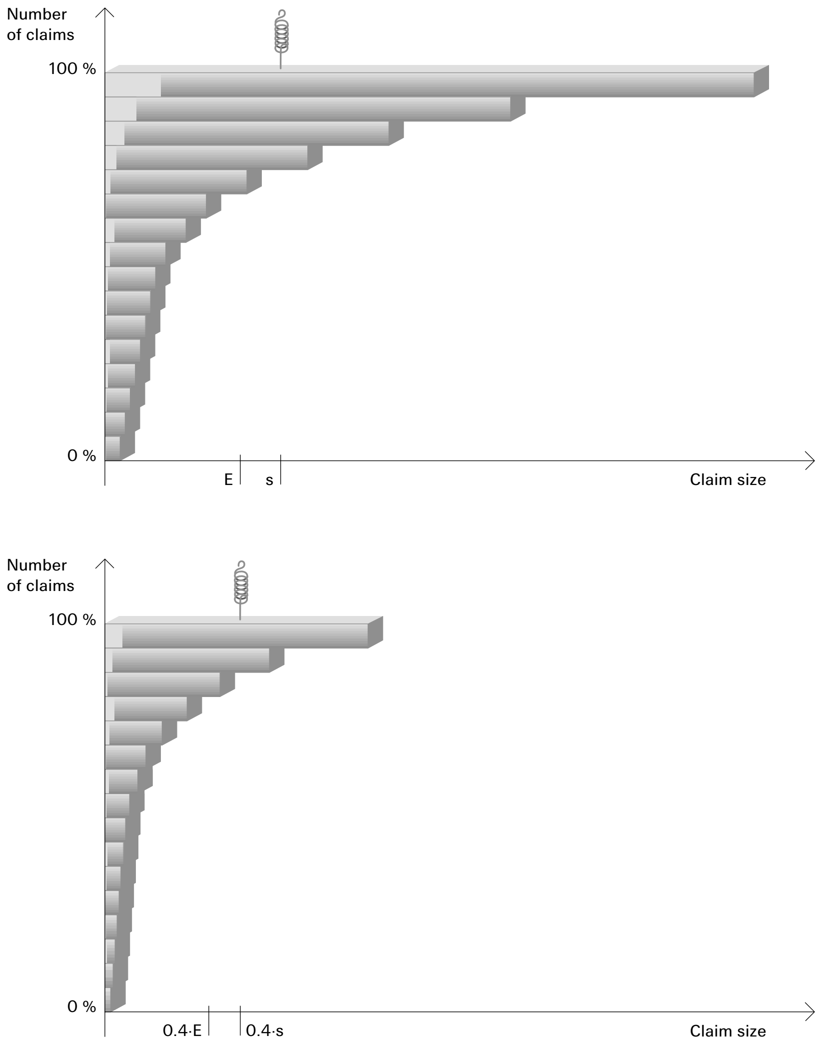

Figure 5

A quota share with a retention of, for example, 40% divides each claim between the primary insurer and the reinsurer in the ratio of 40:60. Figure 5 shows the same claim distribution as Figure 1 once again, ie all possible claims arranged in order of size and piled up, one on top of another. The compressed distribution of the claims in the quota share retention appears beneath them. Each claim is reduced to 40% of its original amount. In the same way the average claim in the retention amounts to 40% of the original average claim E, and the amount of the claim, at which we could suspend the distribution so that it remains in equilibrium, amounts to 40% of s. The quota share reinsurance does not alter the number of claims. Thus, according to section 2, the variance of the loss burden in the retention is calculated as follows:

Variance of the loss burden in the retention = expected number of claims
·expected individual claim
amount·40%
·2·s·40%

In the above equation any quota share retention q can be substituted for 40%. By introducing the abbreviation Z_r (r for retained) for the loss burden in the retention, the variance of the loss burden in the retention can be expressed as

(4.1) $V[Z_r] = \lambda \cdot E \cdot q \cdot 2 \cdot s \cdot q$.

Replacing $E \cdot 2 \cdot s$ with E^2+V in the above equation on the basis of (2.2) yields

(4.2) $V[Z_r] = \lambda \cdot (E^2+V) \cdot q^2$

An excess of loss reinsurance with a deductible d also reduces the variance of the primary insurer. Figure 6 shows how each claim is truncated at an amount of d. The average claim for the primary insurer is thus reduced from E to E_r. The amount of the claim at which the distribution of the truncated claim would be in equilibrium is reduced to s_r. The extent to which s_r is smaller than s and the extent to which E_r, the average claim in the retention, is smaller than E, depends not only on the size of the deductible d, but also on the form of the claim distribution. The following applies to the variance in the retention:

Variance of the loss burden in the retention = expected number of claims
·expected individual claim
amount in the retention
$\cdot 2 \cdot s_r$

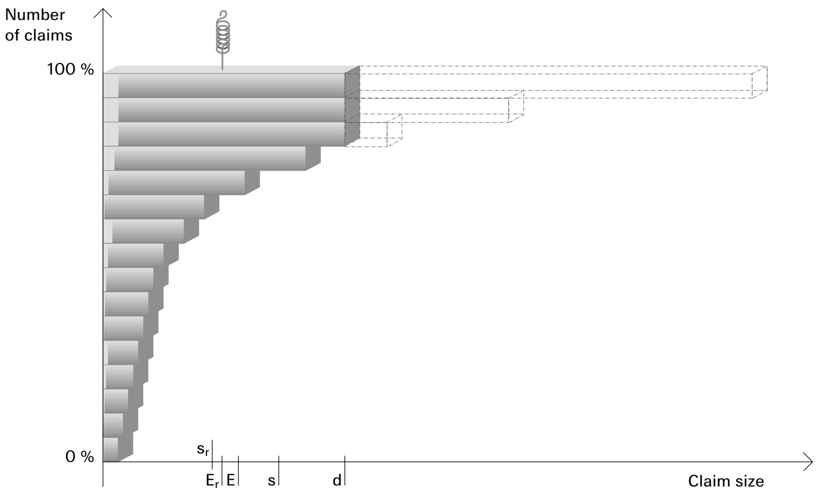

Figure 6

or as an equation

$$(4.3)\ V[Z_r] = \lambda \cdot E_r \cdot 2 \cdot s_r$$

As in (2.2) the expression $E_r \cdot 2 \cdot s_r$ can be replaced by $E_r^2 + V_r$. Then (4.3) becomes

$$(4.4)\ V[Z_r] = \lambda \cdot (E_r^2 + V_r).$$

If the primary insurer protects its portfolio with both an excess of loss and a quota share, then the reduced variance is obtained by replacing the expression E^2+V with $E_r^2+V_r$ in (4.2). This leads to

$$(4.5)\ V[Z_r] = \lambda \cdot (E_r^2 + V_r) \cdot q^2$$

5 Optimal retentions

Every reinsurance treaty covers part of the primary insurer's business, thus reducing the variance of that part. The reduced variances which remain with the primary insurer add up to its total remaining variance. The smaller the retentions of the individual treaties, the smaller the remaining variance becomes and the more balanced the primary insurer's book of business becomes. Should, however, the retentions be too small, the cost of reinsurance will be too high. When the retentions are established at a given total reinsurance price so that the remaining variance cannot be further reduced, then the retentions are optimal. Optimising the retentions at a given reinsurance price requires a clear understanding of what that price actually means.

Every reinsurance premium is the sum of the expected loss burden of the reinsurance treaty and an amount consisting of the reinsurer's loadings for expenses and fluctuations. The expected loss burden is also described as the risk premium, which the primary insurer gets back from the reinsurer in the long term in the form of claims payments. For this reason, it is not a true part of the price at all but rather like a return deposit, which the primary insurer redeems when it presents its claims to the reinsurer. On the other hand, the loadings for expenses and fluctuations remain with the reinsurer in the long term. These alone are the true price for the reinsurance protection. The size of these individual loadings is of no importance. What is important is their combined amount. Rather than the total reinsurance premium, the reinsurance price means the difference between the premium and the risk premium.

Let us now consider two reinsurance treaties A and B, each with its retention (quota share retention or excess of loss deductible), its reinsurance price (fluctuation and expenses loading) and its variance of the loss burden in the retention. The two variances are labelled variance A and variance B. The total variance which remains with the primary insurer from the two treaties is the sum of the individual variances A and B. We must ask whether the total variance from the two treaties can be reduced by a considered choice of the two retentions without changing the total price.

If the retention of treaty A is somewhat increased so that the reinsurance price falls and, at the same time, the retention of treaty B is somewhat reduced so that its reinsurance price increases to the same extent, then the total combined price of A and B remains unchanged. As a result of the increase in A's retention, variance A increases slightly, and variance B is reduced slightly due to the reduction in the retention of B. During this process, the total variance – ie the sum of variance A and variance B – can increase, decrease or remain the same. The primary insurer, of course, is interested in a reduction of the total variance. This occurs when the reduction in variance B is greater than the increase in variance A. In this case, it is advantageous to continue to increase the retention of A and to continue to reduce that of B so that the total price always remains the same but the total variance falls. In the other case, it is necessary to proceed the other way around, ie reduce the retention of A and increase that of B. This strategy can be used until one of the following three situations occurs:

1 Treaty A no longer cedes anything to the reinsurance, the primary insurer retains everything;
2 Treaty B can cede nothing more to the reinsurance, the primary insurer retains nothing;
3 The increase in variance A (due to the increase in the retention of A) and the reduction in variance B (as a result of the reduction in the retention of B) are equal.

In all three cases, a reinsurance solution has been found which leads to the smallest possible total variance at a given total reinsurance price.

To establish retentions for a larger number of reinsurance treaties beyond A and B, we can use exactly the same theory. The total variance is minimal at a given total reinsurance price when with every treaty a small increase in the retention results in the same increase in the variance. In the event of several reinsurance treaties, both of the borderline cases can occur, namely, that some treaties are completely reinsured and others are not reinsured at all.

Customarily, one measures the sum of the loadings for fluctuations and expenses against the expected loss burden of the reinsurer. We denote this ratio with the letter b, ie

$$b = \frac{\text{Reinsurance loadings applied to a quota share treaty}}{\text{Expected loss burden of the reinsurer}}$$

If q is the quota share retention, then the expected loss burden of the reinsurer is equal to

$$\lambda \cdot E \cdot (1\text{-}q),$$

and hence the price is

$$b \cdot \lambda \cdot E \cdot (1\text{-}q).$$

This price falls if q is increased slightly.

According to equation (4.1) the variance of the loss burden in the retention amounts to

$$V[Z_r] = \lambda \cdot E \cdot q \cdot 2 \cdot s \cdot q.$$

It increases if q is increased a little. The derivation of the ratio between the reduction in price and the increase in the variance of the retention is shown in section 10.1; this is denoted as w and amounts to

$$(6.1)\ w = \frac{b \cdot E}{E \cdot 2 \cdot s \cdot q \cdot 2}$$

or

$$(6.2)\ w = \frac{b \cdot E}{(E^2+V) \cdot 2 \cdot q}$$

As shown in section 5, quota share retentions are optimally established when this ratio w is equally high for every quota share treaty.

The relationship (6.2) is applicable in practice as the following numerical example shows. Assuming that a primary insurer considers a 50% retention adequate for its motor liability business, should it likewise retain 50% of its motor own damage business, or more of it or less? The relationship (6.2) can help in the decision-making process. E_H denotes the average liability claim, V_H the variance of the liability claim, b_H the loading for liability, q_H the quota share retention for liability; E_K denotes the average own damage claim, V_K the variance of the own damage claim, b_K the loading for own damage and q_K the quota share retention for own damage yet to be determined. As the ratio w must be equally high for liability and own damage, the following is true

$$\frac{b_H \cdot E_H}{(E_H^2+V_H)\cdot 2\cdot q_H} = \frac{b_K \cdot E_K}{(E_K^2+V_K)\cdot 2\cdot q_K}$$

Solving the equation for q_K gives

$$q_K = q_H \cdot \frac{b_K}{b_H} \cdot \frac{E_K}{E_H} \cdot \frac{E_H^2+V_H}{E_K^2+V_K}$$

If s_H and s_K denote the amount of the claims at which the corresponding distributions would be in equilibrium, then we can also write this last equation as per (2.2):

$$q_K = q_H \cdot \frac{b_K}{b_H} \cdot \frac{s_H}{s_K}$$

This shows clearly that the greater b_K in relation to b_H, the more expensive the motor own damage reinsurance and the less the primary insurer should buy of it. Alternatively, the more unbalanced the motor own damage business is compared to the liability business, the bigger s_K becomes in the denominator compared to s_H in the numerator and the more own damage motor reinsurance the primary insurer should buy. These two facts are self-evident and the equation above merely quantifies this.

The liability figures in Table 2 relate in terms of size to Swiss motor insurance, the euro being the currency unit. Further, due to the length of time motor liability takes to run off, we have calculated discounted claim amounts. This means that future payments are calculated back to their value in the year in which the claims occur at an interest rate of say 5% during the average investment period of around 4.5 years. Based on these assumptions, the non-discounted figures would be approximately 25% higher. Exercise caution in making an assumption about the own damage figures in contrast to the liability amounts, however; they are highly subject to natural catastrophe exposure and vary from insurer to insurer.

	Motor liability	Motor own damage
E	4000	1000
V	$10.2\ 10^8$	$2.2\cdot 10^8$
b	0.1	0.05
q	50%	?

Table 2

If we insert the values from Table 2 into the algebraic expression for the quota share retention q_K, then we obtain the solution

$$q_K = 50\% \cdot \frac{0.05}{0.1} \cdot \frac{1000}{4000} \cdot \frac{4000^2 + 10.2 \cdot 10^8}{1000^2 + 2.2 \cdot 10^8}$$

$$= 29\%.$$

Another use for (6.1) and (6.2) is to establish the line for a surplus reinsurance. Here we consider two claim distributions with the following characteristics:

1 There is a maximum loss which cannot be exceeded (the sum insured or the MPL).
2 Every claim in the one distribution is smaller by the same factor x than the corresponding claim in the other distribution. This holds true, in particular, for the maximum loss, the average claim E and the claim s, at which the distribution would be in equilibrium.

Two such distributions are depicted in Figure 5. Here the factor x is 40%. A comparison of the two claim distributions is summarised in Table 3.

	Claim distribution 1	Claim distribution 2
Maximum loss	M	$M \cdot x$
Average claim	E	$E \cdot x$
Claim amount s	s	$s \cdot x$
Proportional retention q	q_1	q_2

Table 3

Let us now assume the same loading factor b for both claim distributions and insert the values from Table 3 into (6.1):

$$w = \frac{b \cdot E}{E \cdot 2 \cdot s \cdot q_1 \cdot 2}$$

$$= \frac{b \cdot E \cdot x}{E \cdot x \cdot 2 \cdot s \cdot x \cdot q_2 \cdot 2}$$

Solving the equation for q_2, we get

$$q_2 = \frac{q_1}{x}$$

Finally we multiply the proportional retention with the maximum loss. In the case of distribution 1, we get $q_1 \cdot M$ and call this product m; in the case of distribution 2, we get

$$q_2 \cdot M \cdot x = q_1 \cdot M$$
$$= m$$

The proportional retention multiplied by the sum insured or the MPL amounts to the same for both distributions, that is, m. By definition, m is the line of the surplus reinsurance. Therefore a surplus reinsurance yields optimal retentions when the following holds true as in Table 3 for the risks covered: the ratios of the two maximum possible losses, the two corresponding average claims and the two corresponding claim amounts s are the same size. One can also check m against a treaty whose retention has been considered correct. For this we again use the motor liability quota share of Table 2 and a fire risk, for which we use the assumptions summarised in Table 4. Section 10.2 shows how the expected value and the variance of fire losses can be determined.

	Motor liability	Fire
Maximum possible loss		10 000 000
E	4 000	400 000
V	$10.2 \cdot 10^8$	$1.28 \cdot 10^{12}$
b	0.1	0.15
q	50%	?

Table 4

The quota share retention for the fire risk is calculated using (6.2) in exactly the same way as the above for the quota share retention of motor own damage. We obtain

$$q = 50\% \cdot \frac{0.15}{0.1} \cdot \frac{400\,000}{4\,000} \cdot \frac{4000^2 + 10.2 \cdot 10^8}{400\,000^2 + 1.28 \cdot 10^{12}}$$
$$= 5.40\%$$

The line is therefore

$$m = 10\,000\,000 \cdot 0.0540$$
$$= 540\,000.$$

Examining the degree to which a small increase in the deductible d reduces the reinsurance price and increases the variance will show some similarities to section 6.

As in the case of proportional reinsurance, we will denote with the letter c the ratio of the reinsurance loadings to the estimated loss burden of the reinsurer. In this context the expression "reinsurer" refers to the whole reinsurance market. An individual reinsurer would make the loadings factor c dependent on the deductible d, the cover and his share of the treaty. However, the loadings which the market as a whole requires depend mainly on the type of risks covered, ie on the claim distribution. Hence we draw the simplified assumption that c is independent of the size of the deductible d.

In section 10.3, the ratio w between the reduction in the reinsurance price and the increase in the variance in the retention is derived. It is really quite simple:

$$(7.1)\quad w = \frac{c}{2 \cdot d}$$

It is remarkable that in (7.1) the form of the claim distribution has no importance. We can use (7.1) if for example we start with the dedutible of a fire treaty in order to establish the deductible for a natural catastrophe cover. The example in Table 5 shows the quantities required.

Business covered	Reinsurer's loading factor	Deductible
Fire	0.2	500 000
Natural catastrophes	1.0	d

Table 5

From (7.1) we obtain

$$d = 500\,000 \cdot 1/0.2$$

$$= 2\,500\,000.$$

8 Combinations of proportional reinsurance and excess of loss reinsurance

Let us assume a portfolio is protected by an excess of loss reinsurance with a deductible d and for the truncated claims there is a quota share treaty with a quota share retention q. How do d and q need to be aligned to offer the best possible protection against fluctuations? To explore this, we increase the deductible d slightly, which reduces the price of the excess of loss and increases the variance of the truncated claims. Further, we reduce the quota share retention q to the extent that the price increase of the proportional cover becomes as high as the price reduction of the excess of loss cover. As a consequence the variance in the proportional retention will fall. We can differentiate among three cases:

1 The reduction in variance following the reduction of the quota share retention q is smaller than the increase in the variance following the increase of the deductible. In this case, the primary insurer's total variance increases. In other words, changing d and q causes deterioration in the protection. Therefore, it is better to reduce d and to increase q.
2 Alternatively, the reduction in variance following the reduction of the quota share retention q is greater than the increase in the variance following the increase of the deductible. In this case, the primary insurer's total variance becomes smaller; changing d and q has resulted in improved protection. In this case, continuing to increase d and to reduce q is worthwhile.
3 The reduction in variance following the reduction of the quota share retention q is the same size as the increase in the variance following the increase of the deductible. In this case, the primary insurer's total variance remains the same; altering d and q does not result in any further improvement. This combination is optimal.

By using the following expressions
d for the deductible,
E_r for the average claim of the truncated distribution at a level of d and
s_r for the amount of the claim at which the truncated distribution would be in equilibrium,
we can deduce that the optimal retention is as follows:

$$(8.1)\ d_o = \frac{2 \cdot E_r \cdot s_r}{E \cdot \frac{b}{c} - (E - E_r)}$$

The derivation of this is shown in section 10.4.

(8.1) is interpreted as follows. Pure non-proportional reinsurance is the most favourable when w is so small that the deductible d, which has been calculated in accordance with (7.1), is greater than d_o. If according to (7.1) $d < d_o$, then it is preferable to establish $d = d_o$ and to increase w by proportional reinsurance. To do this, we solve the equation (8.2) for q. The derivation of (8.2) is shown in section 10.5.

$$(8.2)\ w = \frac{c}{2 \cdot d_o \cdot q}$$

With regard to the ratio between the loading factors in the denominator in (8.1), b/c, the more expensive proportional reinsurance is compared to non-proportional, the smaller the optimal retention d_o becomes and the more favourable it is to buy non-proportional protection instead of proportional. This is, again, self-evident. We can show that $d_o = 0$ if $c \leq b$ (see [2]). The case where non-proportional reinsurance is less expensive than proportional reinsurance is rather rare, but the consequences are clear. If each retention $d > d_o$, pure non-proportional reinsurance is the best option for any value of w. On the other hand, if c is considerably greater than b, ie when non-proportional reinsurance is much more expensive than proportional, then d_o either becomes very large or does not exist at all. In this case, every d determined in accordance with (7.1) is smaller than d_o and the pure proportional reinsurance solution is the best one.

To obtain a numerical example where d_o exists, we supplement assumptions about motor liability (Table 2) with the following claim distribution information:

- The probability that a discounted claim exceeds EUR 200,000 (ie a non-discounted 250,000) is 0.8%.
- For discounted claims over 200,000 the distribution follows a Pareto distribution with a parameter value of 3 (details on the Pareto distribution can be found in [3]).
- The loading factor for non-proportional reinsurance amounts to c = 0.3.

Section 10.6 shows that according to these assumptions the result is $d_o = 669{,}449$. As was noted in Table 2's comment, the mean of 4,000 and the variance of $10.2 \cdot 10^8$ correspond to discounted values. With non-discounted values 25% higher, the non-discounted optimal deductible, at which proportional reinsurance is still unnecessary, becomes 25% higher than d_o or EUR 836,811. The retention established in the reinsurance treaty is, of course, not discounted.

Table 6 shows the optimal quota shares q and deductibles to selected values of w.

w	Quota share q	Deductible d discounted	Deductible d non-discounted	Non-discounted value·q
$2 \cdot 10^{-8}$	100%	7 500 000	9 375 000	9 375 000
10^{-7}	100%	1 500 000	1 875 000	1 875 000
$2 \cdot 10^{-7}$	100%	750 000	937 500	937 500
$2.24065 \cdot 10^{-7}$	100%	669 449	836 811	836 811
$3 \cdot 10^{-7}$	74.69%	669 449	836 811	625 014
$4 \cdot 10^{-7}$	56.02%	669 449	836 811	468 782

Table 6

A further combination of proportional and non-proportional retentions occurs in property reinsurance. We frequently come across surplus reinsurance where the retention is protected by two excesses of loss: one, an excess of loss per risk, and the other, an excess of loss per event for natural catastrophes. There is also an optimal combination in this case, except that now there are three retentions which need to be aligned, ie the two deductibles of the excess of loss reinsurances and the line of the surplus reinsurance. For a numerical example, we will add the following to the assumptions on fire risks given in Tables 4 and 5:

- The maximum possible loss of 10,000,000 is the line of the surplus reinsurance which we will possibly reduce. However, we are not considering risks with a maximum loss below 10,000,000.
- The average loss degree for fire claims amounts to 4%. Hence the average fire loss in the retention of the surplus reinsurance is 400,000.
- The exposure curve given in Appendix 1 is applicable to the fire risks.
- The loading factor for excess of loss reinsurance per risk amounts to $c_F = 0.2$.
- On average there are $\lambda_F = 100$ fire losses per year.
- The frequency of storm losses amounts to $\lambda_S = 1/25$, ie on average there is a storm loss once every 25 years.
- No storm loss exceeds 100 million.
- If there is a storm loss the probability that that loss does not exceed an amount x is calculated by 1−10 million/(10 million +x) (so the extent of the storm losses is depicted by a Pareto distribution with a parameter 1).
- The loading factor for the catastrophe excess of loss reinsurance amounts to $c_S = 1$.

(7.1) holds true for both the deductibles d_F (fire) and d_S (storm). Therefore

$$\frac{c_F}{d_F} = \frac{c_S}{d_S}$$

or

$$(8.3)\quad d_S = d_F \cdot \frac{c_S}{c_F}$$

On the other hand, as is shown in section 10.7, both deductibles must still fulfil one condition which is closely related to (8.1). We can write (8.1) differently as

$$(8.4)\quad \frac{2 \cdot E_r \cdot s_r \cdot c}{d_o} - (E \cdot b - c \cdot (E - E_r)) = 0$$

Because we are now considering two different claim distributions at the same time, we will extend the previous notation and denote the average loss of the fire loss distribution truncated at d_F with E_{Fr}, the average loss of the catastrophe loss distribution truncated at d_S with E_{sr}, and the claim amounts at which the truncated distributions would be in equilibrium with s_{Fr} and s_{Sr}. For the two deductibles to be optimally combined, the left-hand side of (8.4), is for the one deductible, eg d_F, smaller than 0 and for the other, in this case d_S, greater than 0. The fire deductible is thus higher in this combination than it would be if we were to establish it separately from the catastrophe deductible, and the catastrophe deductible becomes smaller. The derivation in 10.7 shows that the two expressions must counterbalance one another as follows:

$$(8.5)\ \lambda_F \cdot \left(\frac{2 \cdot E_r \cdot s_{Fr} \cdot c_F}{d_F} - E_F \cdot b + c_F \cdot (E_F - E_{Fr})\right) + \lambda_S \cdot \left(\frac{2 \cdot E_{Sr} \cdot s_{Sr} \cdot c_S}{d_S} - E_S \cdot b + c_S \cdot (E_S - E_{Sr})\right) = 0$$

According to the previously made assumptions d_F = 3,080,294 and d_S = 15,401,472 (derivation in 10.7). These values are similarly interpreted as in (8.1): pure non-proportional reinsurance is optimal as long as w is so small that from (7.1) higher values result than d_F and d_S. On the other hand, if w is so large that the deductibles determined according to (7.1) are smaller than d_F and d_S, then it is more favourable to switch to proportional reinsurance. For this purpose, using (8.6) or (8.7) we determine a proportional retention q, which we multiply by the line of the surplus reinsurance, in this example 10 million.

$$(8.6)\ w = \frac{c_S}{2 \cdot d_S \cdot q}$$

$$(8.7)\ w = \frac{c_F}{2 \cdot d_F \cdot q}$$

Some numerical examples are given in Table 7.

w	Quota share q	Line q·10 000 000	Fire deductible	q·fire deductible	Storm deductible	q·storm deductible
$2 \cdot 10^{-8}$	100%	10 000 000	5 000 000	5 000 000	25 000 000	25 000 000
$3.2464429 \cdot 10^{-8}$	100%	10 000 000	3 080 294	3 080 294	15 401 472	15 401 472
10^{-7}	32.46%	3 246 000	3 080 294	999 863	15 401 472	4 999 318
$2 \cdot 10^{-7}$	16.23%	1 623 000	3 080 294	499 932	15 401 472	2 499 659
$3 \cdot 10^{-7}$	10.82%	1 082 000	3 080 294	333 288	15 401 472	1 666 439
$4 \cdot 10^{-7}$	8.12%	812 000	3 080 294	250 120	15 401 472	1 250 600

Table 7

Taking the last example, the figures in Table 7 can be interpreted as follows: when the ratio w between the reduction in price and the increase in variance amounts to $4 \cdot 10^{-7}$, then the optimal line of the surplus reinsurance is 812,000; the optimal deductible of the excess of loss, which protects the retention of the surplus reinsurance against large single claims, amounts to 250,120 and the optimal deductible of the excess of loss, which protects the retention of the surplus reinsurance against natural catastrophes, amounts to 1,250,600.

For every part of the primary insurer's portfolio, the ratio w between the increase in price and the reduction in variance determines an optimal deductible and an optimal quota share retention or a line of a surplus reinsurance. They are optimal in the sense that they reduce the variance which still remains with the primary insurer more effectively than any other combination of proportional and non-proportional reinsurance at the same price. The mathematical relations between w and the quota shares, lines of the surplus reinsurances and deductibles can be found in (6.1), (6.2), (7.1), (8.1), (8.2), (8.3), (8.5), (8.6) and (8.7). We can calculate the reinsurance prices and the variances in the retention on the basis of the average number of claims and the claim distribution of each part of the primary insurer's portfolio. The total price which the primary insurer spends on reinsurance is the sum of all the reinsurance prices, and the total variance in the retention is the sum of the individual variances (for readers with a knowledge of probability theory: we assume that the individual parts of the portfolio are independent). By using Chebyshev's inequality (see section 3) the probability of losing a large amount of capital can be estimated from the total variance. For its part, this probability is a decision tool for choosing w and hence the total reinsurance programme. In order to illustrate these relations, let us use the motor liability and property portfolios from section 8.

Table 8 supplements Table 6. For our motor liability example it shows – for some selected values of w, in addition to the optimal quota share q and the non-discounted value of the deductible d – the accompanying reinsurance price and the variance in the retention. While the expected number of claims plays no role in determining retentions, we do use it to calculate the price and variance. Let us assume $\lambda = 1{,}000$. The assumptions made in section 10.6 form the basis of the numerical calculations. Equations (10.10) and (10.16) are used to determine the price, and (4.5) and (10.17), the variance.

w	Quota share q	Deductible d Non-discounted value	Price	Variance
$2 \cdot 10^{-8}$	100%	9 375 000	171	$10.189 \cdot 10^{11}$
10^{-7}	100%	1 875 000	4 267	$9.507 \cdot 10^{11}$
$2 \cdot 10^{-7}$	100%	937 500	17 067	$8.653 \cdot 10^{11}$
$3 \cdot 10^{-7}$	74.69%	836 811	117 239	$4.713 \cdot 10^{11}$
$4 \cdot 10^{-7}$	56.02%	836 811	187 920	$2.651 \cdot 10^{11}$

Table 8

Calculating the reinsurance price and variance in the retention for the property insurance example in Table 7 leads to Table 9. The calculations are based on the assumptions of section 10.7. The exposure table in Appendix 1 is used to determine the price of the excess of loss per risk and the Pareto model is used to determine the price of the catastrophe cover. The sum of these two prices is the figure which is given in Table 9. The variance of the excess of loss per risk is found using (4.3), (4.4), (4.5) and (10.6). The variance of the catastrophe excess of loss is found using the Pareto model (as in section 10.7) and (4.3), (4.4) and (4.5). The variance in Table 9 is the sum of the variance of the excess of loss per risk and the excess of loss per event.

w	Quota share q	Fire deductible	Storm deductible	Price	Variance
$2 \cdot 10^{-8}$	100%	5 000 000	25 000 000	1 217 253	$1010.911 \cdot 10^{11}$
10^{-7}	32.46%	3 080 294	15 401 472	4 886 075	$62.881 \cdot 10^{11}$
$2 \cdot 10^{-7}$	16.23%	3 080 294	15 401 472	5 514 974	$15.720 \cdot 10^{11}$
$3 \cdot 10^{-7}$	10.82%	3 080 294	15 401 472	5 724 607	$6.987 \cdot 10^{11}$
$4 \cdot 10^{-7}$	8.12%	3 080 294	15 401 472	5 829 230	$3.935 \cdot 10^{11}$

Table 9

We can obtain Table 10 from Tables 8 and 9, by adding the individual prices and the individual variances to the total price and to the total variance respectively in each line. The greater the ratio w, the greater will be the total price and the smaller the total variance. The smaller the variance becomes, the smaller the probability of losing a certain amount of capital, the equity share capital of the company, for example. We can calculate this almost exactly by using probability theory, however we can also estimate it simply by using Chebyshev's basic inequality, as shown in section 3. If we select a capital amount of 15 million, we get a Chebyshev bound for the probability of losing 15 million by dividing the corresponding variance by 15 million^2.

w	Total price	Total variance	Probability
$2 \cdot 10^{-8}$	1 217 424	$1021.100 \cdot 10^{11}$	45.38%
10^{-7}	4 890 342	$72.388 \cdot 10^{11}$	3.22%
$2 \cdot 10^{-7}$	5 532 041	$24.373 \cdot 10^{11}$	1.08%
$3 \cdot 10^{-7}$	5 841 846	$11.700 \cdot 10^{11}$	0.52%
$4 \cdot 10^{-7}$	6 017 150	$6.586 \cdot 10^{11}$	0.29%

Table 10

Figure 7 shows Chebyshev's probability as a function of the reinsurance price. The higher the price, the smaller the probability of losing capital of 15 million. If the primary insurer is prepared to pay a price of, for example, 5 million for the reinsurance protection, then he accepts a probability of losing 15 million of not more than 2.5%.

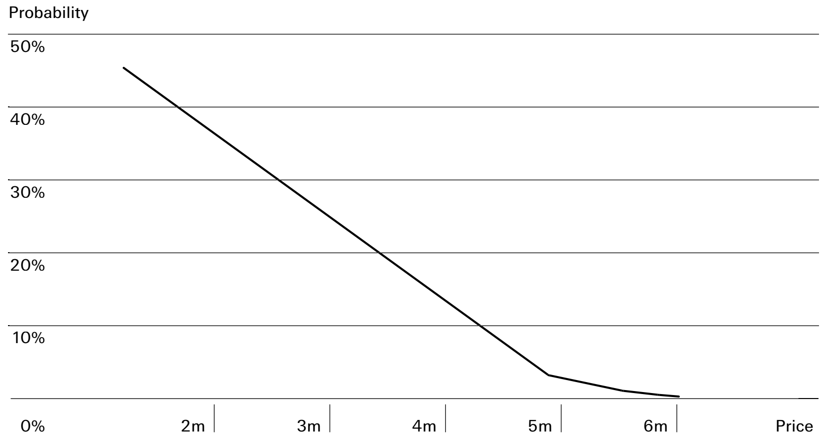

Figure 7

Figure 8 shows the ratio w as a function of the reinsurance price. If the primary insurer decides on a reinsurance price of eg 5 million on the basis of Figure 7, then we can determine the corresponding w from Figure 8. In our case, we get $w = 1.364 \cdot 10^{-7}$. Hence the optimal reinsurance programme is established which is available for 5 million:

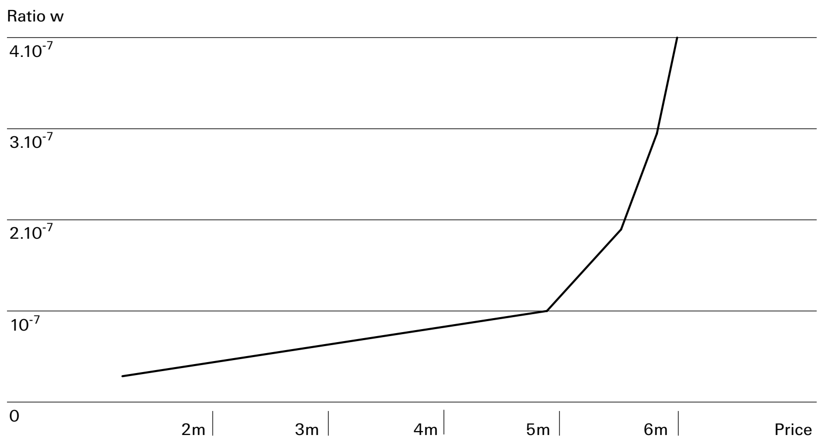

Figure 8

Deductible of motor liability
The discounted value of the deductible is 1,099,707 according to (7.1) and, as such, is higher than d_o. Hence proportional reinsurance is not necessary. The deductible of the excess of loss treaty corresponds to the non-discounted value and is therefore 25% higher than the discounted value, ie 1,374,634.

Property reinsurance
According to (7.1) the deductible of the cover per risk would be 733,138 and hence smaller than d_F. Proportional reinsurance is therefore necessary. From (8.7) we obtain q = 23.8%. In other words:

Line of the surplus reinsurance:	23.8% of 10 million =	2 380,000
Deductible of the cover per risk:	23.8% of 3 080 294 =	733 110
Deductible of the catastrophe cover:	23.8% of 15 401 472 =	3 665 550

10 Derivations

10.1 The price reduction/increase in variance ratio in the quota share retention

We will use the notation from section 6. A small increase of Δ of the quota share retention q reduces the price by

(10.1) reduction in price = $b \cdot \lambda \cdot E \cdot \Delta$.

The increase in the quota share retention also influences the variance of the loss burden in the retention. The expected number of claims remains the same but the component

$E \cdot q \cdot 2 \cdot s \cdot q$

increases. Figure 9 illustrates by just how much it increases. As Figure 4, it shows a rectangle with a length equal to $2 \cdot s$ and a width of E. The quota share retention q defines a smaller rectangle within the larger rectangle measuring $2 \cdot s \cdot q$ in length and $E \cdot q$ in width. The area of this smaller rectangle is the same size as the component $E \cdot q \cdot 2 \cdot s \cdot q$. It increases by the two lighter strips if we increase the quota share retention by Δ. Strictly speaking, it also increases by the small black rectangle in the top right-hand corner, but this is negligible when Δ is sufficiently small. The areas of the two strips amount to $E \cdot q \cdot \Delta \cdot 2 \cdot s$ and $2 \cdot s \cdot q \cdot \Delta \cdot E$.

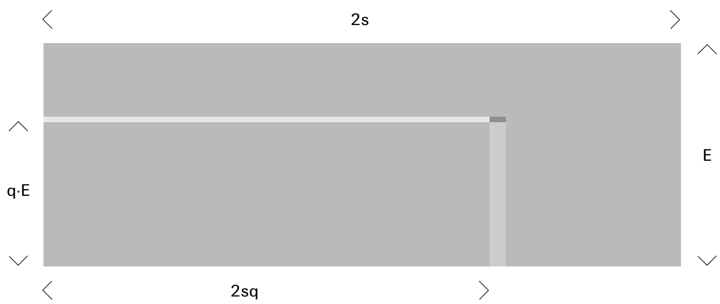

Figure 9

Consequently, due to the small increase in the quota share retention, the variance of the loss burden in the retention increases by

(10.2) increase in variance = $\lambda \cdot (E \cdot q \cdot \Delta \cdot 2 \cdot s + 2 \cdot s \cdot q \cdot \Delta \cdot E)$.

If we calculate the ratio of the reduction in price to the increase in variance, referring to this as w, then the expected number of claims λ and the increase of the quota share retention Δ cancel each other out and what is left is

$$(10.3)\ w = \frac{b \cdot E}{E \cdot 2 \cdot s \cdot q \cdot 2}$$

(10.3) is equation (6.1) from section 6. The denominator in (10.3) can be rewritten with the help of equation (2.2). Then the ratio w between the reduction in price and the increase in variance becomes

$$(10.4)\ w = \frac{b \cdot E}{(E^2 + V) \cdot 2 \cdot q}$$

This is the equation (6.2) in section 6.

10.2 The expected value and variance of fire risks

It is advisable to use loss degree statistics to determine the expected value of fire claims. The loss degree is defined as the ratio between the amount of loss and the MPL or between the amount of loss and the sum insured. Its mean depends on the type of risk under consideration and is in the region of a small percentage of the MPL or sum insured.

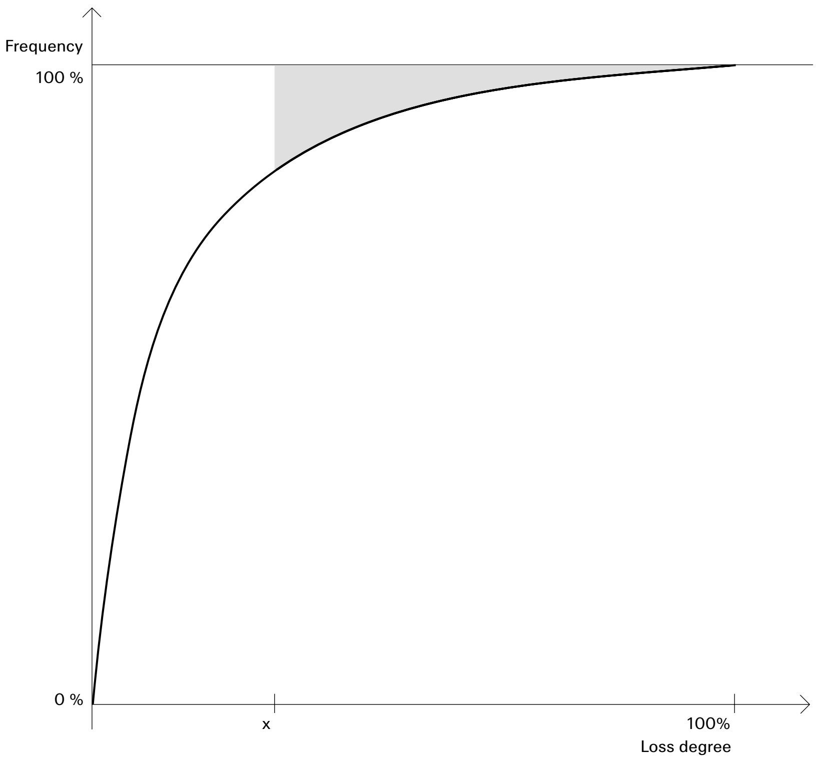

Figure 10

Figure 10 illustrates a loss degree distribution. The area bordered by the vertical axis, the horizontal line at a height of 100% and the curve itself, is the same size as the expected loss degree (more information on these relations can be found in [3]). The shaded area to the right of the loss degree x indicates by how much on average the loss degree exceeds the value x. Dividing this area by the expected loss degree – ie by the total area above the curve – yields as a ratio the average loss degree in excess of x as a portion of the expected loss degree. Figure 11 shows the difference between 1 and this ratio for all loss degrees from 1 to 100%. Referred to as an exposure curve, this type of curve is used to determine the risk premiums when rating excess of loss reinsurance. Exposure tables serve the same purpose. They indicate the value from the exposure curve up to the horizontal line at 100% as a function of the value of the loss degree in table form. An exposure table in Appendix 1 serves as an example.

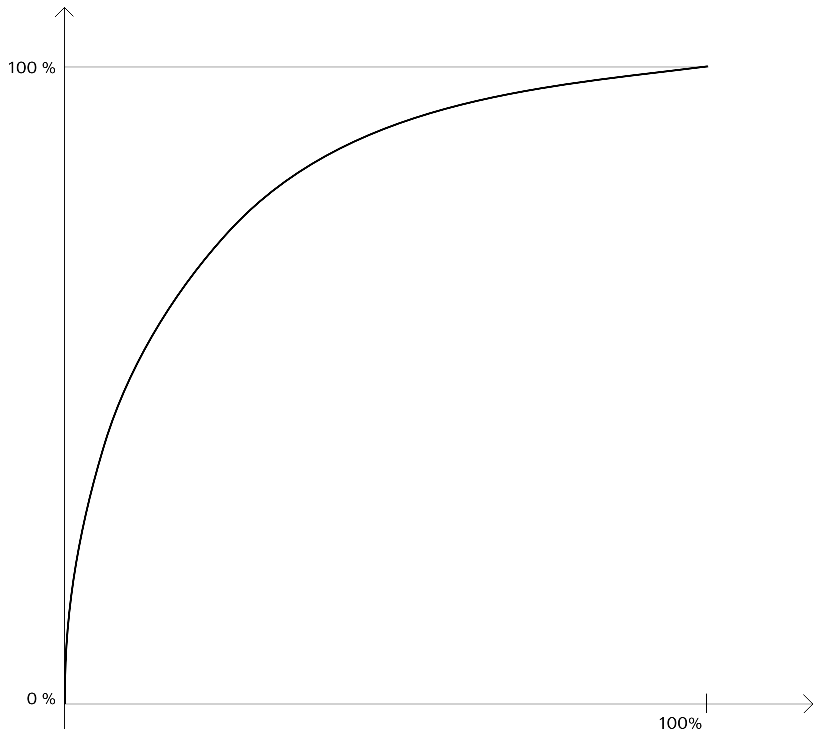

Figure 11

The expected claim of a risk is obtained by multiplying its expected loss degree with the MPL or the sum insured.

Just as the area above the distribution curve represents the expected loss degree in Figure 10, the area above the exposure curve also represents an area which can be interpreted: the loss degree s, at which the distribution would be in equilibrium. This becomes clear when we replace the area above the distribution curve of Figure 10 with narrow vertical strips of any width d. Figure 12 shows an example where d = 10%. The lower side of these strips forms a "flight of stairs" which draws closer to the loss degree distribution the smaller d is. The length of each strip depends on the distribution. For the strip between loss degrees of 30% and 40%, for example, it amounts to 1-F(0.35). Its area is therefore equal to 0.1·(1-F(0.35)). The distribution is then in equilibrium at the loss degree s when the total of all the strip areas, multiplied by their distance from the vertical axis, is the same size as the whole area E multiplied by s, ie

$$(10.5)\quad 0.05\cdot 0.1\cdot(1-F(0.05))+0.15\cdot 0.1\cdot(1-F(0.15))+\ldots\ldots+0.95\cdot 0.1\cdot(1-F(0.95)) = s\cdot E$$

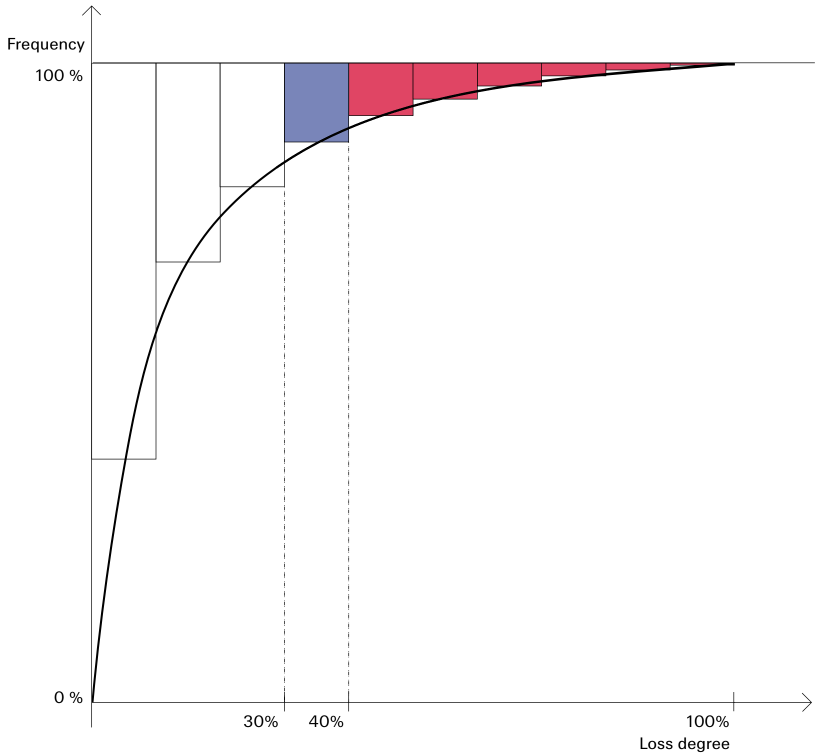

Figure 12

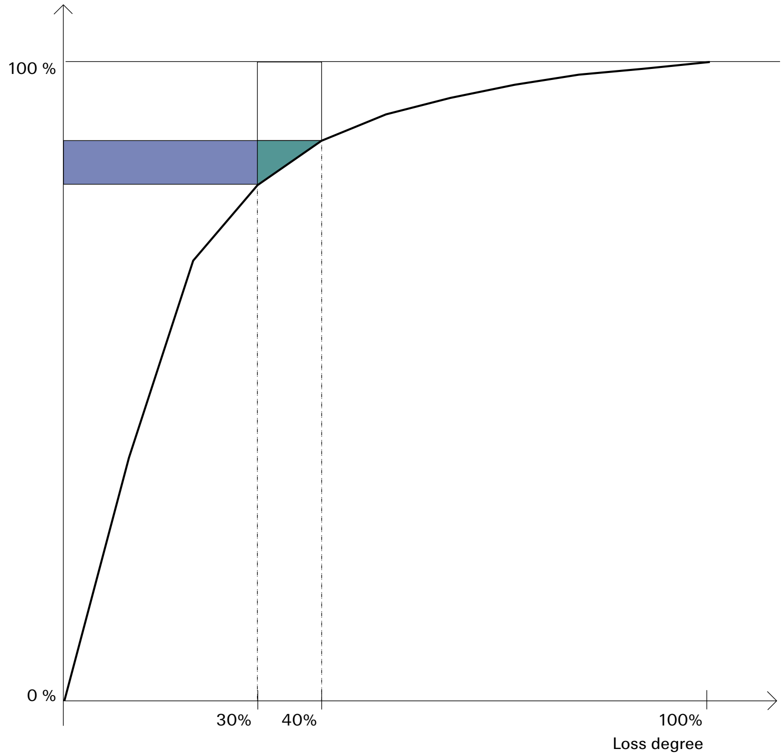

Figure 13

Figure 13 shows the exposure curve which relates to the "flight of stairs" in Figure 12. We can calculate the area above the curve which is formed by the blue rectangle and the green triangle. At a loss degree of 40%, the distance of the exposure curve to the horizontal line at a height of 100% amounts to just as much as the red area in Figure 12, divided by the total area, ie by the expected loss degree. At a loss degree of 30%, the distance of the exposure curve to the horizontal line at a height of 100% is somewhat greater, namely by the ratio of the blue area in Figure 12, divided by the total area, that is by

$$\frac{0.1 \cdot (1-F(0.35))}{E}$$

Therefore, the area of the blue rectangle in Figure 13 with a length of 0.3 is equal to

$$0.3 \cdot \frac{0.1(1-F(0.35))}{E}$$

The area of the green triangle in Figure 13 amounts to

$$0.5 \cdot 0.1 \cdot \frac{0.1(1-F(0.35))}{E}$$

Consequently, the sum of the areas of the blue rectangle and the green triangle is equal to

$$0.35 \cdot \frac{0.1(1-F(0.35))}{E}$$

In exactly the same way, we can calculate other areas above the exposure curve and see that they add up to the left-hand side of (10.5), divided by E, ie to s. By using (2.2) we obtain from E and s the variance of the loss degree.

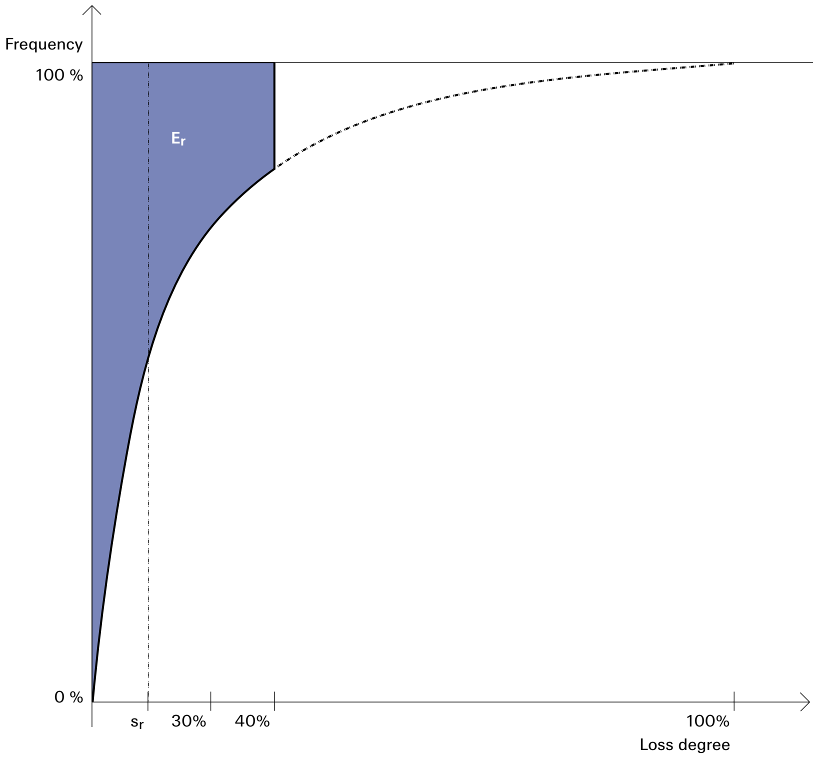

Figure 14

In the same way we can obtain the expected loss degree and the variance of a loss degree distribution, which is truncated at a deductible, eg 40%. This is depicted in Figures 14 and 15. The loss degree distribution in Figure 14 comes from Figure 12 by truncating at the loss degree of 40%. The blue area above the curve represents the average truncated loss degree E_r. The truncated distribution is now no longer in equilibrium at loss degree s, but at the smaller value of s_r. To find this, we need to consider only the vertical strips below the retention 40% in (10.5), ie

$$(10.6)\ 0.05\cdot 0.1\cdot(1-F(0.05)) + ... + 0.35\cdot 0.1\cdot(1-F(0.35)) = s_r\cdot E_r$$

Figure 15 shows the same blue rectangle and the same green triangle as in Figure 13. We determine in the same way the rest of the dark area in Figure 15. Thus, the area over the exposure curve, which is bound to the left by the vertical axis and above by the horizontal line, intersecting the curve at the value of the deductible, is the same size as the left-hand side of (10.6), divided by E. When determining s_r in a specific case, we can use these relations between the exposure curve, E, E_r, and s_r with, for example, a spreadsheet programme.

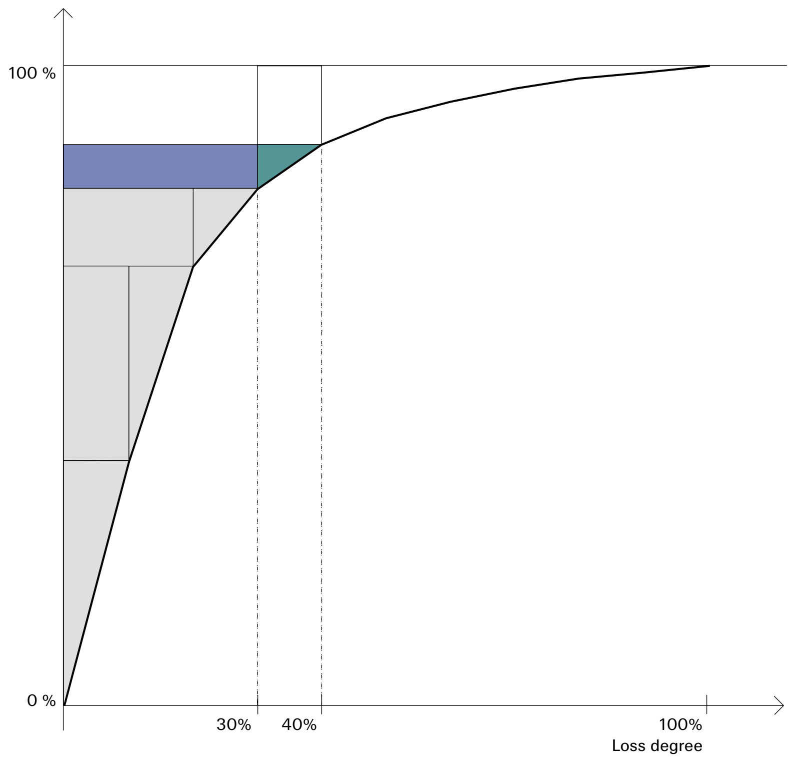

Figure 15

10.3 The price reduction/increase in variance ratio in the excess of loss retention

Figure 16 includes

d the deductible
H the portion of the number of claims, which on average exceed the deductible. If we multiply H by the average number of claims λ, we get the average number of excess claims.
E_r the average claim of the distribution truncated at a height of d
s_r the amount of the claim at which the truncated distribution would be in equilibrium
Δ small amount, eg EUR 1, by which the retention d is to be increased.

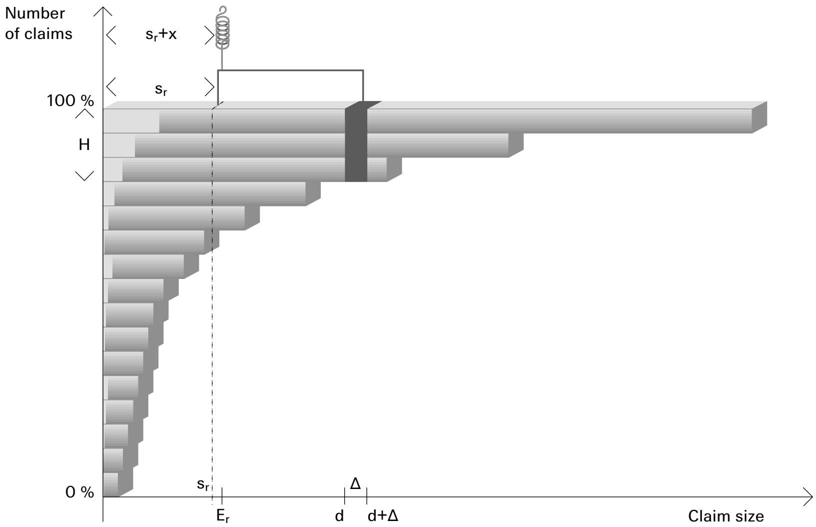

Figure 16

The expected number of losses for the reinsurer amounts to λ·H. If the primary insurer carries Δ himself of each excess loss, then the expected loss burden of the reinsurer reduces by λ·H·Δ, so the reinsurance price becomes smaller by:

(10.7) Reduction in price = $\lambda \cdot H \cdot \Delta \cdot c$

At the same time, the average claim of the truncated distribution increases by H·Δ. In Figure 16, the strip H·Δ is dark. In order to calculate by how much the variance increases, we need to know how far the strip H·Δ moves the amount of claim s_r to the right. This increase is represented by x. The truncated distribution is in equilibrium if, after the increase in the deductible to d+Δ, it is suspended at a claim amount of s_r +x. To the left of the new suspension point the weight E_r hangs at a distance of x, to the right is the weight H·Δ, at a distance of $d-s_r-x+0.5\cdot\Delta$. The following must hold true for equilibrium to prevail

$$x \cdot E_r = (d - s_r - x + 0.5 \cdot \Delta) \cdot H \cdot \Delta$$

$$= (d - s_r) \cdot H \cdot \Delta - x \cdot H \cdot \Delta + 0.5 \cdot \Delta \cdot H \cdot \Delta$$

Δ and x are very small. Multiplied together, their product is negligible. In practice, the above equation becomes

$$x \cdot E_r = (d - s_r) \cdot H \cdot \Delta$$

or if we wish to determine x

$$x = \frac{(d - s_r) \cdot H \cdot \Delta}{E_r}$$

The variance of the loss burden in the retention amounts to $\lambda \cdot E_r \cdot 2 \cdot s_r$ before the retention increase of Δ according to equation (4.3). After the increase, it amounts to $\lambda \cdot (E_r + H \cdot \Delta) \cdot 2 \cdot (s_r + x)$ or, multiplied out,

$$\lambda \cdot E_r \cdot 2 \cdot s_r + \lambda \cdot H \cdot \Delta \cdot 2 \cdot s_r + \lambda \cdot E_r \cdot 2 \cdot x + \lambda \cdot H \cdot \Delta \cdot 2 \cdot x$$

The first term of the sum in this expression is the variance before the increase. The last term becomes negligibly small as it contains the product of the small quantities Δ and x. The noticeable increase in the variance consists of the two summands in the middle. If we substitute the expression $(d - s_r) \cdot H \cdot \Delta$ for $E_r \cdot x$ in the second, then the increase in the variance becomes

(10.8) increase in the variance = $\lambda \cdot 2 \cdot \Delta \cdot H \cdot d$.

Almost everything is cancelled out in the ratio of the reduction in the reinsurance price, $\lambda \cdot H \cdot \Delta \cdot c$, and the increase in the variance. What remains is quite simple, ie

(10.9) $w = \frac{c}{2 \cdot d}$

(10.9) is equation (7.1) from section 7.

10.4 The optimal combination of proportional reinsurance and excess of loss

We will use the notation established in 10.3. Figure 17 shows how the expected claim, ie the area above the distribution curve, is shared between the primary insurer and the reinsurer when a quota share and an excess of loss simultaneously assume part of the loss burden. The quota share retention is q. The green area amounts to (1-q)·E and represents the quota share reinsurance's portion of the expected claim. If λ is the expected number of claims, then the reinsurance price for assuming this portion is λ· (1-q)·E·b. The sum of the blue and red areas indicates the portion of the expected claim which remains with the quota share retention. The deductible d would truncate each claim at the level of d if there were no quota share. The sum of the areas to the left of d, ie the bright green, red and blue, therefore depict the expected truncated claim E_r. The claims in the quota share retention are truncated by the deductible q·d. Therefore the blue area represents $q \cdot E_r$ and the red area $q \cdot (E-E_r)$. The red area is the portion of the expected claim which the excess of loss assumes. The price for this amounts to $\lambda \cdot q \cdot (E-E_r) \cdot c$. The total price for the reinsurance is the sum of the prices for the quota share and excess of loss:

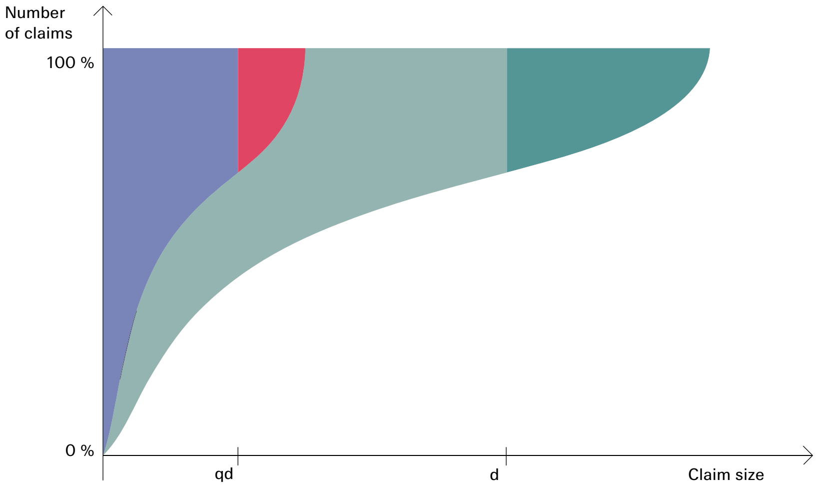

Figure 17

$$(10.10)\ \text{Price} = \lambda \cdot [(1-q) \cdot E \cdot b + q \cdot (E-E_r) \cdot c]$$

Now we increase the deductible somewhat and, at the same time, reduce the quota share retention by just that amount so that the total price remains unchanged. If we call the increase in the deductible Δ_1, then the price reduction of the excess of loss amounts to $\lambda \cdot H \cdot \Delta_1 \cdot c$ according to (10.7). Of this, $\lambda \cdot H \cdot \Delta_1 \cdot c \cdot q$ is allotted to the quota share retention. We denote the reduction in the quota share retention by Δ_2. From (10.10) and (10.1), it follows that the corresponding price increase amounts to

$$(10.11)\ \lambda\cdot(E\cdot b - c\cdot(E-E_r))\cdot\Delta_2$$

The portion q of the price reduction of the excess of loss and the increase in price of the quota share are equally high when

$$\lambda\cdot H\cdot\Delta_1\cdot c\cdot q = \lambda\cdot(E\cdot b - c\cdot(E-E_r))\cdot\Delta_2$$

By solving for Δ_2 we get

$$(10.12)\ \Delta_2 = \frac{H\cdot\Delta_1\cdot c\cdot q}{E\cdot b-c\cdot(E-E_r)}$$

According to (10.8) as a consequence of the deductible increase, the variance in the retention of the excess of loss increases by $\lambda\cdot 2\cdot\Delta_1\cdot H\cdot d$. In the quota share retention, the deductible increase Δ_1 is reduced to $\Delta_1\cdot q$ and d is reduced to $d\cdot q$. The variance thus increases here by $\lambda\cdot 2\cdot\Delta_1\cdot H\cdot d\cdot q\cdot q$. The variance in the quota share retention, however, falls by $\lambda\cdot E_r\cdot q\cdot\Delta_2\cdot 4\cdot s_r$ according to (10.2). Overall, the variance remains the same if

$$(10.13)\ \lambda\cdot 2\cdot\Delta_1\cdot H\cdot d\cdot q\cdot q = \lambda\cdot E_r\cdot q\cdot\Delta_2\cdot 4\cdot s_r$$

or by inserting (10.12) and simplifying

$$d = \frac{2\cdot E_r\cdot s_r\cdot c}{E\cdot b-c\cdot(E-E_r)}$$

It is not clear from the start whether there is in fact a deductible d, which fulfils the above equation. The right-hand side of the equation is an expression actually determined by d (s_r and E_r are dependent on d), which can become higher or lower than d depending on the size of b and c. A deductible which fulfils this equation is denoted as d_o. If we simplify this using c, we get

$$(10.14)\ d_o = \frac{2\cdot E_r\cdot s_r}{E\cdot\frac{b}{c}-(E-E_r)}$$

(10.14) is the same equation as (8.1).

10.5 Derivation of equation (8.2)

10.11 shows by how much the price increases when the quota share retention q decreases by Δ_2, or vice versa, by how much the price decreases when q is increased by Δ_2. The corresponding increase in variance can be determined by applying (10.2) to the retention of an excess of loss: E is replaced by E_r, s by s_r and Δ by Δ_2. After simplifying with λ and Δ_2, the ratio between price reduction and increase in variance can thus be expressed by

$$(10.15)\quad w = \frac{b \cdot E - c \cdot (E - E_r)}{E_r \cdot 2 \cdot s_r \cdot q \cdot 2}$$

(10.15) holds true for every combination of excess of loss and quota share, in particular when taking the optimal deductible d_o for the excess of loss. In this case (10.14) is fulfilled and we may multiply (10.15) by (10.14) and solve for w. The result is (8.2).

10.6 Setting d_o by using a Pareto distribution

Let us assume that the claim distribution in Figure 18 is not known for small claims (smaller than u). However, it has the form of a Pareto distribution to the right of the claim amount u. Therefore the following holds true

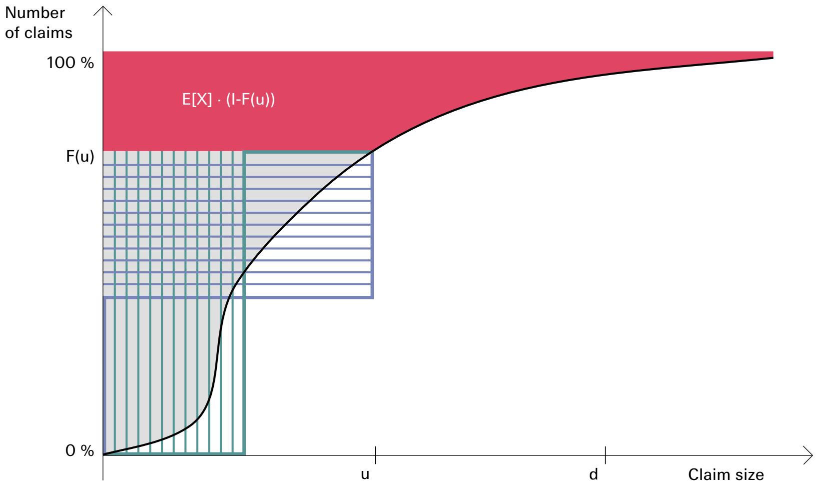

Figure 18

$$1-F(x) = \left(\frac{u}{x}\right)^{\alpha} \cdot (1-F(u))$$

Hence, the conditional claim distribution, given that the individual claim X exceeds the amount u, is a Pareto distribution with a distribution function

$$P(X \leq x) = 1 - \left(\frac{u}{x}\right)^{\alpha}$$

The following holds true for this Pareto distribution on the assumption $\alpha>2$:

$$E[X] = u \cdot \frac{\alpha}{\alpha-1}$$

and

$$E[X^2] = u^2 \cdot \frac{\alpha}{\alpha-2}$$

For the individual claim Y=X-(X-d)+ truncated at d, we have

$$E[Y] = \frac{u}{\alpha-1} \cdot \left(\alpha - \left(\frac{u}{d}\right)^{\alpha-1}\right)$$

and

$$E[Y^2] = \frac{u^2}{\alpha-2} \cdot \left(\alpha - 2 \cdot \left(\frac{u}{d}\right)^{\alpha-2}\right)$$

Let E represent the expected value of the claim distribution of Figure 18 and E_r the expected value of the distribution truncated at d. Then

$$(10.16)\ E_r = E - E[X] \cdot (1-F(u)) + E[Y] \cdot (1-F(u)).$$

The variance of V_r of the distribution truncated at d can be calculated by using E, E_r and the variance V of the untruncated claim distribution, ie by

$$(10.17)\ E_r^2 + V_r = E^2 + V - E[X^2] \cdot (1-F(u)) + E[Y^2] \cdot (1-F(u))$$

To replace the numerator in (8.1) in a similar way to (2.2) with $E_r^2+V_r$ is to determine d_o.

There is no Pareto distribution for arbitrary values of E, V and F(u), which fulfils (10.16) and (10.17. For the distribution to exist, the inequalities (10.18), (10.19) and (10.20) must be fulfilled.

$$(10.18)\ E[X] \cdot (1-F(u)) \leq E$$

This follows directly from Figure 18. The total area between the vertical line at 0, the horizontal line at a height of 1 and the distribution curve is the same size as E and therefore certainly at least as big as the red area.

If we replace the part of the distribution curve in Figure 18 between 0 and u with the green line (so that the green-shaded area is the same size as the grey area), then we get a distribution which corresponds to the original distribution for x>u, with the same expected value E but a smaller variance. This variance plus the square of E is equal to the expression on the left-hand side of the inequality sign in (10.19)

$$(10.19)\quad E[X^2]\cdot(1-F(u)) + \frac{[E-E[X]\cdot(1-F(u))]^2}{F(u)} \leq E^2+V$$

If we replace the original distribution below u with the blue line, so that the blue-shaded area is the same size as the grey area, we get a distribution which corresponds to the original distribution for x>u, with the same expected value E but a greater variance. The expression on the left of (10.20) is equal to this variance plus the square of the expected value.

$$(10.20)\quad E[X^2]\cdot(1-F(u)) + [E-E[X](1-F(u))]\cdot u \leq E^2+V$$

Using the values b=0.1, c=0.3, α=3, 1-F(u) = 0.008 and d =669 449
we get E_r=3 928.5972, $E_r^2+V_r$ = 844 797 981.6237, so that (8.1) is fulfilled.

10.7 The optimal combination of proportional reinsurance with two excesses of loss

Specific fire and storm losses would fall under the two claim distributions, and by supplementing the 10.4 notation, we can differentiate between them:

- λ_F the average number of fire losses
- d_F the fire deductible
- H_F the portion of the number of fire losses which on average exceed the deductible. If we multiply H_F by the average number of fire losses λ_F, we get the average number of fire excess losses.
- E_{Fr} the average loss of a fire loss distribution truncated at a height of d_F
- s_{Fr} the amount of loss, at which the truncated fire loss distribution would be in equilibrium
- Δ_F a small amount by which the deductible d_F is to be increased
- λ_S the average number of storm events
- d_S the storm deductible
- H_S the portion of the number of storm losses which on average exceed the deductible. Every storm event loss is made up of small individual storm losses. Multiplying H_S by the average number of storm events λ_S, yields the average number of storm excess losses.
- E_{Sr} the average loss of a storm loss distribution truncated at a height of d_s
- s_{Sr} the amount of loss at which the truncated storm loss distribution would be in equilibrium

Δ_S a small amount by which the retention d_S is to be increased
q the quota share retention
Δ_2 a small reduction in the quota share retention

(8.3) holds true for the two excesses of loss. From this, it follows that

$$(10.21)\ \Delta_S = \Delta_F \cdot \frac{c_S}{c_F}$$

An increase in the fire deductible by Δ_F results in a price reduction of the fire excess of loss by $\lambda_F \cdot H_F \cdot \Delta_F \cdot c_F$ similar to the derivation in 10.4. At the same time due to (10.21), the storm deductible increases, resulting in a reduction of the storm price of $\lambda_S \cdot H_S \cdot \Delta_S \cdot c_S$. The total reduction in price, which results from the increases in the fire and storm deductibles, amounts (if we replace Δ_S using (10.21)) to

$$\lambda_F \cdot H_F \cdot \Delta_F \cdot c_F + \lambda_S \cdot H_S \cdot \Delta_F \cdot \frac{c_S}{c_F} \cdot c_S$$

From this price reduction, the portion q is allotted to the quota share retention, ie

$$q \cdot \Delta_F \cdot (\lambda_F \cdot H_F \cdot c_F + \lambda_S \cdot H_S \cdot \frac{c_S}{c_F} \cdot c_S)$$

Now we reduce the quota share retention by Δ_2 and thus increase the price of the proportional cover in a similar way to (10.11) by

$$\lambda_F \cdot (E_F \cdot b - c_F \cdot (E_F - E_{Fr})) \cdot \Delta_2 + \lambda_S \cdot (E_S \cdot b - c_S \cdot (E_S - E_{Sr})) \cdot \Delta_2$$

The total reinsurance price remains unchanged if the price reduction (as a result of the deductible increases) and the price increase (as a result of the reduction of the quota share retention) are the same. In this case

$$(10.22)\ q \cdot \Delta_F \cdot (\lambda_F \cdot H_F \cdot c_F + \lambda_S \cdot H_S \cdot \frac{c_S}{c_F} \cdot c_S) = \lambda_F \cdot (E_F \cdot b - c_F \cdot (E_F - E_{Fr})) \cdot \Delta_2 + \lambda_S \cdot (E_S \cdot b - c_S \cdot (E_S - E_{Sr})) \cdot \Delta_2$$

The variance in the retentions of the two excesses of loss becomes greater due to the increase in deductibles and it decreases following the reduction of the quota share retention. Overall it remains the same, if, similar to (10.13) the following holds true

$$\lambda_F \cdot 2 \cdot \Delta_F \cdot H_F \cdot d_F \cdot q^2 + \lambda_S \cdot 2 \cdot \Delta_F \cdot \frac{c_S}{c_F} \cdot c_S \cdot H_S \cdot d_S \cdot q^2 = \lambda_F \cdot E_{Fr} \cdot q \cdot \Delta_2 \cdot 4 \cdot s_{Fr} + \lambda_S \cdot E_{Sr} \cdot q \cdot \Delta_2 \cdot 4 \cdot s_{Sr}$$

After simplifying and replacing d_S using (8.3), what remains is

$$(10.23) \lambda_F \cdot \Delta_F \cdot H_F \cdot d_F \cdot q + \lambda_S \cdot \Delta_F \cdot \frac{c_S}{c_F} \cdot H_S \cdot d_F \cdot \frac{c_S}{c_F} \cdot q = \lambda_F \cdot E_{Fr} \cdot \Delta_2 \cdot 2 \cdot s_{Fr} + \lambda_S \cdot E_{Sr} \cdot \Delta_2 \cdot 2 \cdot s_{Sr}$$

If we multiply (10.23) with c_F and divide it by d_F, then the left-hand side of the equation becomes as large as the left-hand side of (10.22). Therefore, the right-hand sides are also equal. If we subtract the one from the other and divide by Δ_2 we get

$$(10.24)\ \lambda_F \cdot \left(\frac{2 \cdot E_{Fr} \cdot s_{Fr} \cdot c_F}{d_F} - E_F \cdot b + c_F \cdot (E_F - E_{Fr})\right) + \lambda_S \cdot \left(\frac{2 \cdot E_{Sr} \cdot s_{Sr} \cdot c_S}{d_S} - E_S \cdot b + c_S \cdot (E_S - E_{Sr})\right) = 0$$

(10.24) is identical to (8.5).

We can work out the numerical values of d_F and d_S most easily by using a spreadsheet. If we use the exposure table from Appendix 1, assuming m = 10 million and E_F = 400,000 and the assumptions of section 8, we get the following values

To deductible d_F = 3 080 294
E_{Fr} becomes E_{Fr} = 315 865.8

According to (10.2)

$$\frac{s_{Fr} \cdot E_{Fr}}{E} = [0.005 \cdot 0.2206 + 0.015 \cdot (0.2614 - 0.2206) + 0.025 \cdot (0.2995 - 0.2614) + ... + 0.295 \cdot (0.7830 - 0.7743)$$

$$+ (0.3 + 0.0080294 \cdot 0.5) \cdot (0.7913 - 0.7830) \cdot 0.80294] \cdot 10^7$$
$$= 685\,200.76$$

The last summand in the square brackets is established by interpolation. If we had d_F = 3,000,000, then it would amount to 0.295 · (0.7913 − 0.7830). However, for d_F = 3,080,294 it amounts to only

$$(0.3 + 0.0080294 \cdot 0.5) \cdot (0.7913 - 0.7830) \cdot \frac{3\,080\,294 - 3\,000\,000}{3\,100\,000 - 3\,000\,000}$$

$$= 0.3040147 \cdot (0.7913 - 0.7830) \cdot 0.80294$$

In the square brackets, the loss degrees between 0 and 1 are used rather than currency units. Therefore the brackets are multiplied by the line of the surplus reinsurance, 10^7.

$$2 \cdot E_{Fr} \cdot s_{Fr} = 2 \cdot 685200.76 \cdot 400000$$
$$= 5.481606 \cdot 10^{11}.$$

The storm losses are, according to the assumptions of section 8, represented as excess losses of a Pareto distribution, whereby the retention amounts to 10^7, the cover to 10^8 and the Pareto parameter to 1. The numerical values of E_{Sr} and $2 \cdot E_{Sr} \cdot s_{Sr}$ can most easily be calculated by using the equations from [3], Appendix 1 (although a different notation is used there, namely $E[(X\text{-}OP)^2]$ instead of $2 \cdot E_{Sr} \cdot s_{Sr}$).

For d_S = 15,401,472 we get

$$E_S = 10^7 \cdot \ln(11)$$
$$= 23978953$$

$$E_{Sr} = 10^7 \cdot \ln\left(\frac{10^7 + 15401472}{10^7}\right)$$

$$= 10^7 \cdot \ln(2.5401472)$$

$$= 9322220$$

and

$$2 \cdot E_{Sr} \cdot s_{Sr} = 2 \cdot 10^{14} \cdot \left(\frac{10^7 + 15401472}{10^7} - 1 - \ln\left(\frac{10^7 + 15401472}{10^7}\right)\right)$$
$$= 1.21585 \cdot 10^{14}$$

With $\lambda_F = 100$, $\lambda_S = 0.04$, $b=0.15$, $c_F = 0.2$ and $c_S = 1$ (10.24) is fulfilled.

Appendix 1: Exposure table

The exposure table shown below has been calculated based on losses at office buildings, schools, hospitals etc with an MPL of more than CHF 50,000 (according to 1984 values).

d: deductible as a percentage
P: reinsurance risk premium as a percentage of the risk premium

d	P	d	P	d	P	d	P	d	P
1	77.94	21	30.91	41	13.92	61	6.05	81	2.67
2	73.86	22	29.70	42	13.35	62	5.82	82	2.50
3	70.05	23	28.54	43	12.80	63	5.61	83	2.33
4	66.50	24	27.44	44	12.27	64	5.41	84	2.15
5	63.18	25	26.39	45	11.76	65	5.21	85	1.98
6	60.08	26	25.37	46	11.27	66	5.03	86	1.79
7	57.17	27	24.40	47	10.80	67	4.85	87	1.61
8	54.46	28	23.47	48	10.34	68	4.68	88	1.42
9	51.91	29	22.57	49	9.91	69	4.52	89	1.23
10	49.53	30	21.70	50	9.49	70	4.36	90	1.05
11	47.29	31	20.87	51	9.10	71	4.21	91	0.87
12	45.19	32	20.06	52	8.72	72	4.05	92	0.69
13	43.22	33	19.28	53	8.36	73	3.90	93	0.53
14	41.36	34	18.53	54	8.01	74	3.75	94	0.45
15	39.61	35	17.80	55	7.69	75	3.61	95	0.38
16	37.96	36	17.10	56	7.38	76	3.46	96	0.30
17	36.39	37	16.42	57	7.08	77	3.30	97	0.23
18	34.91	38	15.76	58	6.80	78	3.15	98	0.15
19	33.51	39	15.13	59	6.54	79	2.99	99	0.08
20	32.18	40	14.51	60	6.29	80	2.83	100	0.00

Appendix 2: Index of symbols

α	Pareto parameter
b	Loading factor of the proportional reinsurance
c	Excess of loss reinsurance loading factor
d	Deductible
d_o	Optimal deductible from combining proportional and excess of loss reinsurance
Δ	Small change in the deductible or quota share retention
E	Expected value of the individual claim
E_r	Expected value of the truncated individual claim
E[..]	Expected value
F(x)	Distribution function of the claim size
H	Portion of the number of claims which on average exceed the deductible
λ	Expected number of claims per year
M	Maximum loss
m	Line of a surplus treaty
q	Proportional retention, quota share retention
s	Amount of the claim at which the individual claim distribution would be in equilibrium
s_r	Amount of the claim at which the truncated individual claim distribution would be in equilibrium
u	Smallest claim of a Pareto distribution
V	Variance of the individual claim
V_r	Variance of the truncated individual claim
V[Z]	Variance of the loss burden
w	Ratio between price reduction and increase in variance in the retention
X	Claim size of a Pareto distribution
Y	Claim size of a truncated Pareto distribution
Z	Loss burden

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Author:
Hans Schmitter
Group Product Management Casualty,
Chief Underwriting Office

Programming:
Pamela Hall

Editing and production:
Technical Communications,
Chief Underwriting Office

Translation:
Swiss Re Language Services,
Communications & Human Resources division

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